1 INTRODUCTION

The definition of continua for real-world materials is presented, as are elementary examples through which basic ideas of mechanics evolved.

1.1 THE OBJECTIVE OF THIS COURSE

Our objective is to learn how to formulate problems in mechanics and how to reduce vague questions and ideas into precise mathematical statements, as well as to cultivate a habit of questioning, analyzing, designing, and inventing in engineering and science.

Let us consider a few questions. Suppose an airplane is flying above us. The wings must be under strain in order to support the passengers and freight. How much strain are the wings subjected to? If you were flying a glider, and an anvil cloud appeared, the thermal current would carry the craft higher. Dare you fly into the cloud? Have the wings sufficient strength? Ahead you see the Golden Gate Bridge. Its cables support a tremendous load. How does one design such cables? The cloud contains water and the countryside needs that water. If the cloud ciples that underlie these differential equations and boundary conditions. Although it would be nice to solve these equations once they are formulated, we shall not become involved in discussing their solutions in detail. Our objective is formulation: the formal reduction of general ideas to a mathematical form.

1.2 APPLICATIONS TO SCIENCE AND TECHNOLOGY

The mathematical approach taken in this book will be aimed at serving science and technology. I want the applications to be apparent to the student; hence, the examples and the problems to be solved are often stated in terms of scientific research or engineering design. A person's frame of mind with regard to designing and inventing things, devices, methods, theories, and experiments can be strengthened by constant practice—by forming a habit.

1.3 WHAT IS MECHANICS?

Mechanics is the study of the motion (or equilibrium) of matter and the forces that cause such motion (or equilibrium). Mechanics is based on the concepts of time, space, force, energy, and matter. A knowledge of mechanics is needed for the study of all branches of physics, chemistry, biology, and engineering.

1.4 A PROTOTYPE OF A CONTINUUM: THE CLASSICAL DEFINITION

The classical concept of a continuum is derived from mathematics. We say that the real number system is a *continuum*. Between any two distinct real numbers there is another distinct real number, and therefore, there are infinitely many real numbers between any two distinct real numbers. Intuitively, we feel that *time* can be represented by a real number system t and that a three-dimensional *space* can be represented by three real number systems x, y, z. Thus, we identify *time* and *space* together as a four-dimensional continuum.

Extending the concept of a continuum to *matter*, we speak of a continuous distribution of matter in space. This may be best illustrated by considering the concept of *density*. Let the amount of matter be measured by its *mass*, and let us assume that a certain matter permeates a certain space $\mathcal{V}_0$, as in Fig. 1.1. Let us consider a point P in $\mathcal{V}_0$ and a sequence of subspaces $\mathcal{V}_1$, $\mathcal{V}_2$, . . . , converging on P:

$$\mathcal{V}_n \subset \mathcal{V}_{n-1}, \qquad P \in \mathcal{V}_n, \qquad (n = 1, 2, \ldots). \tag{1.4–1}$$

Let the volume of $\mathcal{V}_n$ be V_n and the mass of the matter contained in $\mathcal{V}_n$ be M_n. We form the ratio M_n/V_n. Then if the limit of M_n/V_n exists as $n \to \infty$ and $V_n \to 0$, the

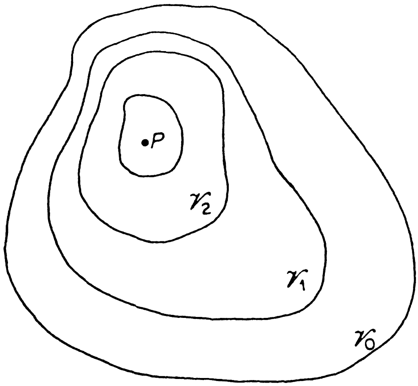

Figure 1.1 A sequence of spatial domains converging on P.

limiting value is defined as the *density of the mass distribution at the point P* and is denoted by $\rho(P)$:

$$\rho(P) = \lim_{\substack{n\to\infty \\ V_n\to 0}} \frac{M_n}{V_n}. \tag{1.4–2}$$

If the density is well defined everywhere in $\mathcal{V}_0$, the mass is said to be *continuously distributed.*

A similar consideration can be used to define the density of momentum, the density of energy, and so on. *A material continuum is a material for which the densities of mass, momentum, and energy exist in the mathematical sense. The mechanics of such a material continuum is continuum mechanics.*

This is the usual definition of a material continuum. However, if we adhere rigorously to it, it will be of no use to science and technology, because the set of real-world systems satisfying such a definition is empty. For example, no gas would satisfy Eq. (1.4–2) when V_n becomes smaller than the mean free path. And no fluid would satisfy the equation when V_n becomes atomic sized. No polycrystalline metal or fiber composite structure, no ceramic, and no polymer plastic can meet this requirement: no living organism, no tissue of any animal, no single cell, and no cell aggregate can either.

1.5 OUR DEFINITION OF A CONTINUUM

We shall define a material as a continuum in a way similar to the classical approach presented in the preceding section, except that the size of V_n will be bounded below and the material particles will not be required to have a one-to-one isomorphism

P. As $n \to \infty$, *the limit of* V_n *tends to a finite positive number* ω. Let the mass of the material enclosed in $\mathcal{V}_n$ be M_n. The sequence of the ratios M_n/V_n is said to have a *limit* ρ *with an acceptable variability* ϵ if

$$\left| \rho - \frac{M_n}{V_n} \right| < \epsilon$$

as $n \to \infty$. The quantity ρ is then said to be the *density of the material at* P *with an acceptable variability* ϵ *in a defining limit volume* ω.

We define the momentum of the material particles per unit volume and the energy per unit volume similarly, each associated with an acceptable variability and a defining volume. Later (see Sec. 1.6), we shall deal with the force acting on a surface of a material body, and it would be necessary to consider whether a limit of force per unit area exists at any point on the surface with an acceptable variability in a defining limit area. If it does exist, then the limit is called the *traction* or *stress*, and the collective entity of tractions in every orientation of the surface is called the *stress tensor*. Further, in Chap. 5 we shall consider the change of spacing between particles and define the *strain tensor*. The existence of strain components will be associated with an acceptable variability and a defining limit length.

If, with a clear understanding of acceptable variabilities and defining limit lengths, areas, and volumes, the density, momentum, energy, stress, and strain can be defined at every point in the space $\mathcal{V}_0$, and if they are all continuous functions of spatial coordinates in $\mathcal{V}_0$, then we say that the material in $\mathcal{V}_0$ is a *continuum*.

If a material is a continuum in the classical sense, then it is also a continuum in our sense. For a classical continuum the acceptable variability and the defining limit length, area, and volume are zero.

In other books on continuum mechanics, the authors say or imply that to decide whether continuum mechanics is applicable to science and technology is a matter for the experimenters in each discipline to decide. I say instead that every experimenter knows that the classical theory does not apply; hence, it is the responsibility of the theorist to refine the theory to fit the real world. Our approach does fit many fields of science and technology; the need to specify acceptable variabilities and defining dimensions is the price we pay.

1.6 THE CONCEPT OF STRESS IN OUR DEFINITION OF A CONTINUUM

Consider a material B occupying a spatial region V (Fig. 1.2). Imagine a closed surface S within B, and consider the interaction between the material outside S and that in the interior. Let ΔS be a small surface element on S. Let us draw, from a point on ΔS, a unit vector $\boldsymbol{\nu}$ normal to ΔS, with its direction pointing outward from the interior of S. Then we can distinguish the two sides of ΔS according to the direction of $\boldsymbol{\nu}$. Let the side to which this normal vector points be called the positive side. Consider the part of material lying on the positive side. This part exerts a force $\Delta \mathbf{F}$ on the other part, which is situated on the negative side of the