1 Schemes and Coherent Sheaves

1.1 Presheaves and Sheaves

Let X be a topological space. A *presheaf* $\mathcal{F}$ of sets on X consists of the following data:

(a) For every nonempty open subset U of X, we have a set $\mathcal{F}(U)$ whose elements are called *sections* of $\mathcal{F}$ over U.

(b) For every inclusion $V \subset U$ of nonempty open subsets of X, we have a map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, called the *restriction*.

These data satisfy

(i) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$.

(ii) if $W \subset V \subset U$ are open subsets of X, then $\rho_{UW} = \rho_{VW}\rho_{UV}$.

For any section $s \in \mathcal{F}(U)$ and $V \subset U$, we often denote $\rho_{UV}(s)$ by $s|_V$. Elements in $\mathcal{F}(U)$ are often denoted by (s, U) in order to make the open set U explicit in the notation. We make the convention that $\mathcal{F}(\varnothing) = \{0\}$ for any presheaf of sets $\mathcal{F}$.

Define the category of open subsets of X so that its objects are nonempty open subsets of X, and for any two objects U and V, define

$$\mathrm{Hom}(V, U) = \begin{cases} \varnothing & \text{if } V \not\subset U, \\ \{V \hookrightarrow U\} & \text{if } V \subset U. \end{cases}$$

Then a presheaf $\mathcal{F}$ of sets on X is just a contravariant functor from the category of open subsets of X to the category of sets.

Similarly we define a presheaf $\mathcal{F}$ of Abelian groups (resp. rings) on X to be a contravariant functor $\mathcal{F}$ from the category of open subsets of X to the category of Abelian groups (resp. rings). For any presheaf of Abelian groups or rings, we make the convention that $\mathcal{F}(\varnothing) = \{0\}$.

Here are some examples of presheaves:

1. Let X be a topological space and A an Abelian group. For every nonempty open subset U of X, define $\mathcal{F}(U) = A$, and for every inclusion $V \subset U$ of nonempty open subsets, define $\rho_{UV} = \mathrm{id}_A$. Then $\mathcal{F}$ is a presheaf of Abelian groups, called the *constant presheaf* associated to A.

2. Let X be a topological space. For every open subset U of X, define $\mathcal{C}(U)$ to be the ring of complex valued continuous functions on U, and for every inclusion $V \subset U$ of nonempty open subsets, define $\rho_{UV} : \mathcal{C}(U) \to \mathcal{C}(V)$ to be the restriction of functions. Then $\mathcal{C}$ is a presheaf of rings.

3. Let $\pi : X' \to X$ be a continuous map of topological spaces. For every nonempty open subset U of X, define $\mathcal{S}(U)$ to be the set of continuous sections of π over U:

$$\mathcal{S}(U) = \{s : U \to \pi^{-1}(U) | \pi s = \mathrm{id}, \text{ and } s \text{ is continuous}\},$$

and for every inclusion $V \subset U$ of nonempty open subsets, define $\rho_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$ to be the restriction of sections. Then $\mathcal{S}$ is a presheaf of sets.

We say a presheaf $\mathcal{F}$ of sets (resp. Abelian groups, resp. rings) is a *sheaf* if it satisfies the following conditions:

(i) Let $s, t \in \mathcal{F}(U)$ be two sections. If there exists an open covering $\{U_i\}_{i\in I}$ of U such that $s|_{U_i} = t|_{U_i}$ for any i, then $s = t$.

(ii) Suppose $\{U_i\}_{i\in I}$ is an open covering of U and $s_i \in \mathcal{F}(U_i)$ are some sections satisfying $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for any $i, j \in I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for any $i \in I$. (By (i), such s is unique.)

Note that a presheaf $\mathcal{F}$ of Abelian groups is a sheaf if and only if for any open covering $\{U_i\}_{i\in I}$ of any open subset U, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is exact, where the second arrow is

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i),\ s \mapsto (s|_{U_i})$$

and the third arrow is

$$\prod_{i\in I} \mathcal{F}(U_i) \to \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j),\ (s_i) \mapsto (s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}).$$

The last two examples of presheaves given above are sheaves.

A *direct set* is a partially ordered set $(I, \leqslant)$ such that for any $i, j \in I$, there exists a $k \in I$ such that $i, j \leqslant k$. A *direct system* $(A_i, \phi_{ij})_{i\in I}$ of sets consists of a family of sets A_i $(i \in I)$ and maps $\phi_{ij} : A_i \to A_j$ for pairs $i \leqslant j$ such that $\phi_{ii} = \mathrm{id}_{A_i}$ and $\phi_{jk}\phi_{ij} = \phi_{ik}$ whenever $i \leqslant j \leqslant k$. For any $x_i \in A_i$ and $x_j \in A_j$, we say x_i is equivalent to x_j if there exists a $k \geqslant i, j$ such that $\phi_{ik}(x_i) = \phi_{jk}(x_j)$. This defines an equivalence relation on the disjoint union $\coprod_i A_i$ of A_i $(i \in I)$. The *direct limit* $\mathrm{dir.\,lim}_i\, A_i$ of $(A_i, \phi_{ij})_{i\in I}$ is defined to be the set of equivalence classes.

Let X be a topological space and P a point in X. For any two neighborhoods U and V of P, we say $V \leqslant U$ if $U \subset V$. Then the family of neighborhoods of P becomes a direct set with respect to this order. For any presheaf $\mathcal{F}$ on X, define the *stalk* $\mathcal{F}_P$ of $\mathcal{F}$ at P by

$$\mathcal{F}_P = \mathrm{dir.}\lim_{P\in U} \mathcal{F}(U),$$

where the direct limit is taken over the family of neighborhoods of P. So elements of $\mathcal{F}_P$ can be represented by sections of $\mathcal{F}$ over some neighborhoods of P. Two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in $\mathcal{F}_P$ if and only if there exists a neighborhood W of P such that $W \subset U \cap V$ and $s|_W = t|_W$. For any neighborhood U of P, we have a canonical map $\mathcal{F}(U) \to \mathcal{F}_P$. The image of a section $s \in \mathcal{F}(U)$ in $\mathcal{F}_P$ is called the *germ* of s at P and is denoted by s_P.

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of Abelian groups on X. A *morphism of presheaves* $\phi : \mathcal{F} \to \mathcal{G}$ consists of a homomorphism of Abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset U such that for every inclusion $V \subset U$ of open subsets, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \stackrel{\phi(U)}{\to} & \mathcal{G}(U) \\ \rho_{UV}\downarrow & & \downarrow\rho_{UV} \\ \mathcal{F}(V) & \stackrel{\phi(V)}{\to} & \mathcal{G}(V). \end{array}$$

For any point $P \in X$, ϕ induces a homomorphism on stalks $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$. If we regard presheaves of Abelian groups as contravariant functors from the category of open subsets on X to the category of Abelian groups, then a morphism of presheaves is just a natural transformation. Similarly we can define morphisms between presheaves of sets or rings. For any presheaf $\mathcal{F}$, we have the identity morphism $\mathrm{id}_{\mathcal{F}}$. Given two morphisms of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$, we can define their composite $\psi\phi : \mathcal{F} \to \mathcal{H}$ in the obvious way. We thus get the category of presheaves. A morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ is called an *isomorphism* if it has a two-sided inverse, that is, there exists a morphism of presheaves $\psi : \mathcal{G} \to \mathcal{F}$ such that $\psi\phi = \mathrm{id}_{\mathcal{F}}$ and $\phi\psi = \mathrm{id}_{\mathcal{G}}$. This is equivalent to saying that $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for every open subset U. We define morphisms of sheaves as morphisms of presheaves. We thus get the category of sheaves which is a full subcategory of the category of presheaves.

Proposition 1.1.1 Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then ϕ is an isomorphism if and only if the induced map on stalks $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for every $P \in X$.

Proof The " only if " part is obvious. Let's prove the " if " part. Suppose $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is bijective for every $P \in X$. We need to show $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective for every open subset U of X.

Let $s, s' \in \mathcal{F}(U)$ be two sections such that $\phi(s) = \phi(s')$. Then $\phi_P(s_P) = \phi_P(s'_P)$ for any $P \in U$. Since ϕ_P is injective, we have $s_P = s'_P$. So there exists a neighborhood U_P of P contained in U such that $s|_{U_P} = s'|_{U_P}$. Note that $\{U_P\}_{P\in U}$ is an open covering of U. Since $\mathcal{F}$ is a sheaf, we must have $s = s'$. So $\phi(U)$ is injective.

Let (t, U) be a section in $\mathcal{G}(U)$. For any $P \in U$, since $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is surjective, we may find $s_P \in \mathcal{F}_P$ such that $\phi_P(s_P) = t_P$. We may assume s_P is the germ of a section $(s, U_P) \in \mathcal{F}(U_P)$ for some neighborhood U_P of P. Note that $\phi(s, U_P)$ and (t, U) have the same germ at P. Choosing U_P sufficiently small, we may assume $U_P \subset U$ and $\phi(s, U_P) = (t, U)|_{U_P}$. Then for any two points $P, Q \in U$, we have

$$\phi(s, U_P)|_{U_P\cap U_Q} = (t, U)|_{U_P\cap U_Q} = \phi(s, U_Q)|_{U_P\cap U_Q}.$$

By the injectivity of $\phi(U_P \cap U_Q)$ that we have proved above, we must have $(s, U_P)|_{U_P\cap U_Q} = (s, U_Q)|_{U_P\cap U_Q}$. Note that $\{U_P\}_{P\in U}$ form an open covering of U. Since $\mathcal{F}$ is a sheaf, we may find a section $(s, U) \in \mathcal{F}(U)$ such that $(s, U)|_{U_p} = (s, U_P)$ for any $P \in U$. We have $(\phi(s, U))|_{U_P} = (t, U)|_{U_P}$. Since $\mathcal{G}$ is a sheaf, we must have $\phi(s, U) = (t, U)$. So $\phi(U)$ is surjective.

Before going on, we introduce some concepts from the theory of categories. Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ in $\mathcal{C}$ is called a *monomorphism* or *injective* if for any two morphisms $\alpha, \beta : C \to A$ satisfying $f\alpha = f\beta$, we have

$\alpha = \beta$. An epimorphism is defined similarly by reversing the directions of arrows. More precisely, $f : A \to B$ is called an *epimorphism* or *surjective* if for any two morphisms $\alpha, \beta : B \to C$ satisfying $\alpha f = \beta f$, we have $\alpha = \beta$. If a morphism is both injective and surjective, we say it is *bijective*. An *isomorphism* is a morphism with a two-sided inverse. Any isomorphism is bijective. But a bijective morphism may not be an isomorphism.

Let A_i $(i \in I)$ be a family of objects in $\mathcal{C}$. The *direct product* of A_i $(i \in I)$ is an object $\prod_{i\in I} A_i$ together with a family of morphisms $p_i : \prod_{i\in I} A_i \to A_i$ $(i \in I)$ called *projections* with the following universal property: For any object C and any family of morphisms $f_i : C \to A_i$ $(i \in I)$, there exists one and only one morphism $f : C \to \prod_{i\in I} A_i$ such that $p_i f = f_i$ for any i. If the direct product of A_i $(i \in I)$ exists, it is unique up to unique isomorphism, that is, any two direct product of A_i $(i \in I)$ are isomorphic and the isomorphism between them is unique.

The direct sum of A_i $(i \in I)$ is defined similarly as above by reversing the directions of arrows. More precisely, the *direct sum* of A_i $(i \in I)$ is an object $\oplus_{i\in I} A_i$ together with a family of morphisms $k_i : A_i \to \oplus_{i\in I} A_i$ $(i \in I)$ with the following universal property: For any object C and any family of morphisms $f_i : A_i \to C$ $(i \in I)$, there exists one and only one morphism $f : \oplus_{i\in I} A_i \to C$ such that $f k_i = f_i$ for any i. If the direct product of A_i $(i \in I)$ exists, it is unique up to unique isomorphism.

Let $(I, \leqslant)$ be a directed set. A *direct system* $(A_i, \phi_{ij})_{i\in I}$ consists of a family of objects A_i $(i \in I)$ and morphisms $\phi_{ij} : A_i \to A_j$ for pairs $i \leqslant j$ such that $\phi_{ii} = \mathrm{id}_{A_i}$ for any i and $\phi_{jk}\phi_{ij} = \phi_{ik}$ whenever $i \leqslant j \leqslant k$. The *direct limit* of a direct system (A_i, ϕ_{ij}) is an object $\mathrm{dir.\,lim}_i A_i$ together with morphisms $\phi_i : A_i \to \mathrm{dir.\,lim}_i A_i$ $(i \in I)$ satisfying $\phi_j \phi_{ij} = \phi_i$ whenever $i \leqslant j$ and having the following universal property: For any object C and any morphisms $\psi_i : A_i \to C$ $(i \in I)$ satisfying $\psi_j \phi_{ij} = \psi_i$ $(i \leqslant j)$, there exists a unique morphism $\psi : \mathrm{dir.\,lim}_i A_i \to C$ such that $\psi \phi_i = \psi_i$ for any i. If the direct limit exists, it is unique up to unique isomorphism. Let $(A'_i, \phi_{ij})_{i\in I}$ be another direct system. A *morphism* from (A_i, ϕ_{ij}) to (A'_i, ϕ_{ij}) is a family of morphisms $u_i : A_i \to A'_i$ $(i \in I)$ such that for any $i \leqslant j$, the following diagram commutes:

$$\begin{array}{ccc} A_i & \stackrel{u_i}{\to} & A'_i \\ \phi_{ij} \downarrow & & \downarrow \phi'_{ij} \\ A_j & \stackrel{u_j}{\to} & A'_j. \end{array}$$

It induces a morphism $\mathrm{dir.\,lim}_i u_i : \mathrm{dir.\,lim}_i A_i \to \mathrm{dir.\,lim}_i A'_i$.

An *inverse system* $(A_i, \phi_{ji})_{i\in I}$ consists of a family of objects A_i $(i \in I)$ and morphisms $\phi_{ji} : A_j \to A_i$ for pairs $i \leqslant j$ such that $\phi_{ii} = \mathrm{id}_{A_i}$ for any i and $\phi_{ji}\phi_{kj} = \phi_{ki}$ whenever $i \leqslant j \leqslant k$. The *inverse limit* of an inverse system (A_i, ϕ_{ji}) is an object $\mathrm{inv.\,lim}_i\, A_i$ together with morphisms $\phi_i : \mathrm{inv.\,lim}_i\, A_i \to A_i$ $(i \in I)$ satisfying $\phi_{ji}\phi_j = \phi_i$ whenever $i \leqslant j$ and having the following universal property: For any object C and any morphisms $\psi_i : C \to A_i$ $(i \in I)$ satisfying $\phi_{ji}\psi_j = \psi_i$ $(i \leqslant j)$, there exists a unique morphism $\psi : C \to \mathrm{inv.\,lim}_i\, A_i$ such that $\phi_i\psi = \psi_i$ for any i. A *morphism* from an inverse system $(A_i, \phi_{ji})_{i\in I}$ to an inverse system $(A'_i, \phi_{ji})_{i\in I}$ is a family of morphisms $u_i : A_i \to A'_i$ $(i \in I)$ such that for any $i \leqslant j$, the following diagram commutes:

$$\begin{array}{ccc} A_j & \xrightarrow{u_j} & A'_j \\ {\scriptstyle \phi_{ji}}\downarrow & & \downarrow {\scriptstyle \phi'_{ji}} \\ A_i & \xrightarrow{u_i} & A'_i. \end{array}$$

It induces a morphism $\mathrm{inv.\,lim}_i\, u_i : \mathrm{inv.\,lim}_i\, A_i \to \mathrm{inv.\,lim}_i\, A'_i$. If $(A_i, \phi_{ji})_{i\in I}$ is an inverse system of sets, then $\mathrm{inv.\,lim}_i\, A_i$ is the subset of $\prod_i A_i$ consisting of those elements $(x_i) \in \prod_i A_i$ satisfying $\phi_{ji}(x_j) = x_i$ for any $i \leqslant j$.

A category $\mathcal{C}$ is called an *additive category* if for any objects A, B and C in $\mathcal{C}$, the direct product of A and B exists, $\mathrm{Hom}(A, B)$ is an Abelian group, and the map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),\ (f, g) \mapsto gf$$

is a homomorphism. We call $0 \in \mathrm{Hom}(A, B)$ the *zero morphism*.

Proposition 1.1.2 Let $\mathcal{C}$ be an additive category and let A and B be two objects in $\mathcal{C}$.

(i) Let $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ be the projections. Define $k_1 : A \to A \times B$ to be the unique morphism satisfying $p_1k_1 = \mathrm{id}_A$ and $p_2k_1 = 0$, and define $k_2 : B \to A \times B$ to be the unique morphism satisfying $p_1k_2 = 0$ and $p_2k_2 = \mathrm{id}_B$. Then we have $k_1p_1 + k_2p_2 = \mathrm{id}_{A\times B}$.

(ii) Suppose we have an object P and morphisms $p_1 : P \to A$, $p_2 : P \to B$, $k_1 : A \to P$, $k_2 : B \to P$ such that

$$\begin{aligned} p_1k_1 &= \mathrm{id}_A, \\ p_2k_2 &= \mathrm{id}_B, \\ k_1p_1 + k_2p_2 &= \mathrm{id}_P. \end{aligned}$$

Then P together with the morphisms $p_1 : P \to A$ and $p_2 : P \to B$ is the direct product of A and B, P together with the morphisms $k_1 : A \to P$ and $k_2 : B \to P$ is the direct sum of A and B.

Proof (i) It is easy to verify that

$$p_1(k_1p_1 + k_2p_2) = p_1\mathrm{id}_{A\times B},$$
$$p_2(k_1p_1 + k_2p_2) = p_2\mathrm{id}_{A\times B}.$$

By the universal property of the direct product, we have $k_1p_1 + k_2p_2 = \mathrm{id}_{A\times B}$.

(ii) We have

$$\begin{aligned} p_1k_2 &= p_1\mathrm{id}_P k_2 = p_1(k_1p_1 + k_2p_2)k_2 \\ &= (p_1k_1)(p_1k_2) + (p_1k_2)(p_2k_2) = p_1k_2 + p_1k_2 \\ &= 2p_1k_2. \end{aligned}$$

So $p_1k_2 = 0$. Similarly $p_2k_1 = 0$.

Let's prove (P, k_1, k_2) is the direct sum of A and B and leave to the reader to prove (P, p_1, p_2) is the direct product of A and B. Given any object C and any morphisms $f_1 : A \to C$ and $f_2 : B \to C$, define $f = f_1p_1 + f_2p_2$. It is easy to verify that $fk_1 = f_1$ and $fk_2 = f_2$. If $f' : P \to C$ is a morphism such that $f'k_1 = f_1$ and $f'k_2 = f_2$, then we have

$$\begin{aligned} f' &= f'\mathrm{id}_P = f'(k_1p_1 + k_2p_2) \\ &= (f'k_1)p_1 + (f'k_2)p_2 = f_1p_1 + f_2p_2. \end{aligned}$$

This proves (P, k_1, k_2) has the required universal property.

Let $\mathcal{C}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{C}$. We say a monomorphism $K \to A$ is the *kernel* of f if the composite $K \to A \to B$ is 0, and for any morphism $K' \to A$ such that the composite $K' \to A \to B$ is 0, there exists a unique morphism $K' \to K$ such that the diagram

$$\begin{array}{ccc} K' & & \\ \downarrow & \searrow & \\ K & \to & A \end{array}$$

commutes. We often denote K by $\ker f$ and call it the kernel of f. Similarly we define the *cokernel* of f to be an epimorphism $B \to C$ such that the composite

$A \to B \to C$ is 0, and for any morphism $B \to C'$ such that the composite $A \to B \to C'$ is 0, there exists a unique morphism $C \to C'$ such that the diagram

$$\begin{array}{ccc} B & \to & C \\ & \searrow & \downarrow \\ & & C' \end{array}$$

commutes. We often denote C by coker$f$ and call it the cokernel of f. We define the *image* of f to be the kernel of the cokernel of f, and define the *coimage* of f to be the cokernel of the kernel of f. There exists a canonical morphism coim$f \to$ im$f$ from the coimage to the image such that the diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \uparrow \\ \mathrm{coim} f & \to & \mathrm{im} f \end{array}$$

commutes. For example, when $f : A \to B$ is a morphism in the category of Abelian groups, then

$$\begin{aligned} \mathrm{ker} f &= \{a \in A | f(a) = 0\}, \\ \mathrm{im} f &= \{b \in B | b = f(a) \text{ for some } a \in A\}, \\ \mathrm{coker} f &= B/\mathrm{im} f, \\ \mathrm{coim} f &= A/\mathrm{ker} f, \end{aligned}$$

and the canonical morphism from the coimage to the image is the canonical homomorphism $A/\mathrm{ker}f \to \mathrm{im}f$ (which is an isomorphism).

Let $\mathcal{C}$ be an additive category. A *zero object* 0 in $\mathcal{C}$ is an object such that $\mathrm{Hom}(0,0) = \{0\}$. This is equivalent to saying that the identity morphism of 0 is equal to the zero morphism. For any object X in $\mathcal{C}$, we have $\mathrm{Hom}(X, 0) = \{0\}$ and $\mathrm{Hom}(0, X) = \{0\}$. Zero objects in $\mathcal{C}$ are isomorphic to each other.

An *Abelian category* $\mathcal{C}$ is an additive category with zero objects such that for any morphism f in $\mathcal{C}$, the kernel and cokernel of f exist (and hence the image and coimage of f exist), and the canonical morphism coim$f \to$ im$f$ is an isomorphism.

In an Abelian category, a bijective morphism is an isomorphism. Indeed, if $f : A \to B$ is injective, then the kernel of f is $0 \to A$ and the coimage of f is $\mathrm{id}_A : A \to A$. If $f : A \to B$ is surjective, then the cokernel of f is $B \to 0$ and the image of f is $\mathrm{id}_B : B \to B$. If $f : A \to B$ is bijective, then the canonical morphism coim$f \to$ im$f$ is just $f : A \to B$. Since coim$f \to$ im$f$ is an isomorphism, $f : A \to B$ is an isomorphism.

Suppose $u : A \to B$ is a monomorphism in an Abelian category. We often say A is a *sub-object* of B. Let $B \to C$ be the cokernel of u. We call C the *quotient* of B by A and denote it by B/A.

In an Abelian category, a sequence of morphisms

$$A \xrightarrow{u} B \xrightarrow{v} C$$

is called *exact* if $vu = 0$ and the canonical morphism $\mathrm{coim}u \to \ker v$ is an isomorphism. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a *short exact sequence*. This short exact sequence is called *split* if it is isomorphic to

$$0 \to A \to A \oplus C \to C \to 0,$$

where $A \to A \oplus C$ and $A \oplus C = A \times C \to C$ are the canonical morphisms.

Proposition 1.1.3 Let

$$0 \to A_1 \xrightarrow{i_1} A \xrightarrow{p_2} A_2 \to 0$$

be a short exact sequence in an Abelian category. The following conditions are equivalent.

(i) The above short exact sequence is split.

(ii) There exists a morphism $p_1 : A \to A_1$ such that $p_1 i_1 = \mathrm{id}_{A_1}$.

(iii) There exists a morphism $i_2 : A_2 \to A$ such that $p_2 i_2 = \mathrm{id}_{A_2}$.

Proof (i)⇒(ii) and (i)⇒ (iii) are obvious.

(ii)⇒(i) Consider the morphism $\mathrm{id}_A - i_1 p_1 : A \to A$. We have

$$(\mathrm{id}_A - i_1 p_1) i_1 = i_1 - i_1(p_1 i_1) = 0.$$

Since A_2 is the cokernel of $i_1 : A_1 \to A$, there exists a morphism $i_2 : A_2 \to A$ so that $\mathrm{id}_A - i_1 p_1 = i_2 p_2$. Our assertion then follows from Proposition 1.1.2 (ii).

Similarly one can prove (iii)⇒(i).

Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between Abelian categories. We say F is *additive* if for any objects A and B in $\mathcal{C}$, the map

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

is a homomorphism. We then have

$$F(A \oplus B) \cong F(A) \oplus F(B).$$

Indeed, keeping the notations in Proposition 1.1.2 (i) and applying F to the equalities there, we get

$$\begin{aligned} F(p_1)F(k_1) &= \mathrm{id}_{F(A)}, \\ F(p_2)F(k_2) &= \mathrm{id}_{F(B)}, \\ F(k_1)F(p_1) + F(k_2)F(p_2) &= \mathrm{id}_{F(A\oplus B)}. \end{aligned}$$

So by Proposition 1.1.2 (ii), $(F(A \oplus B), F(k_1), F(k_2))$ is the direct sum of $F(A)$ and $F(B)$. Hence if

$$0 \to A \to B \to C \to 0$$

is a split short exact sequence, then

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is also a split short exact sequence.

Note that the category of Abelian groups is an Abelian category. We leave to the reader to prove the following proposition:

Proposition 1.1.4 Let X be a topological space. Then the category of presheaves of Abelian groups on X is an Abelian category. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of Abelian groups. Then the kernel, cokernel and image of ϕ are the presheaves defined by

$$\begin{aligned} (\ker\phi)(U) &= \ker(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)), \\ (\mathrm{coker}\phi)(U) &= \mathrm{coker}(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)), \\ (\mathrm{im}\phi)(U) &= \mathrm{im}(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)) \end{aligned}$$

for every open subset U of X. The stalks of these presheaves at a point $P \in X$ are given by

$$\begin{aligned} (\ker\phi)_P &= \ker(\phi_P : \mathcal{F}_P \to \mathcal{G}_P), \\ (\mathrm{coker}\phi)_P &= \mathrm{coker}(\phi_P : \mathcal{F}_P \to \mathcal{G}_P), \\ (\mathrm{im}\phi)_P &= \mathrm{im}(\phi_P : \mathcal{F}_P \to \mathcal{G}_P). \end{aligned}$$