CHAPTER 1

Classical Mechanics

1.1 NEWTON'S LAWS, THE ACTION, AND THE HAMILTONIAN

The superposition principle and the probability interpretation determine the mathematical framework of quantum mechanics. But, like Newton's three laws of classical mechanics, these two ideas do not tell us what the observables or how they change with time. For this physics still leans on classical mechanics in the form of the correspondence principle; not just the obvious idea that classical mechanics comes out of quantum mechanics in the appropriate limit, but the deeper idea that classical Hamiltonian mechanics leads to the correct quantum rules.

The equations of motion in quantum mechanics look like the equations of motion in classical mechanics. This resemblance has been a powerful tool for guessing the rules in quantum mechanics. The method, called **canonical quantization**, is based on Hamilton's "canonical" formulation of classical mechanics in terms of p's and q's. There is also a Lagrangian approach, the powerful path integral formulation of quantum mechanics, but even there the underlying postulates are stated in the Hamiltonian formalism.

Canonical quantization is not guaranteed to work—its predictions must be tested experimentally. And there are quantum mechanical ideas like spin and parity that have no close classical analog. Nevertheless it has proved a powerful idea, predicting correctly the behavior of quantum systems and, as a bonus, giving us hope that if the classical theory is consistent, then so is the corresponding quantum theory.

In this chapter I will review very briefly Hamiltonian classical mechanics with a particular view to those features we will use in quantum mechanics.

1.1.1 Newton's Law and Lagrange's Equations

Let us start where one always starts, from Newton's laws. The simplest mechanical systems have vector coordinates $\mathbf{r}_n$, one vector for each particle. The kinetic energy is

$$T = \frac{1}{2}\sum_n m_n \dot{\mathbf{r}}_n^2 = \frac{1}{2}\sum_{n,i} m_n \dot{r}_{n,i}^2 \tag{1.1}$$

where $r_{n,i}$ are the three Cartesian coordinates of the n-th particle. The potential energy $V(\mathbf{r}_n, \dot{\mathbf{r}}_n, t)$ is a function of the coordinates and perhaps the velocities. In the simplest cases V is independent of $\dot{\mathbf{r}}_n$ and also of the time. For these systems the **Lagrangian** is defined as

$$L = T - V \tag{1.2}$$

L is a function of the coordinates and velocities. It depends on the time, since the coordinates and velocities depend on the time. The i-th component of the force

on the n-th particle is

$$F_{n,i} = -\frac{\partial V}{\partial r_{n,i}} = \frac{\partial L}{\partial r_{n,i}} \tag{1.3}$$

while its momentum is

$$p_{n,i} = m_n \dot{r}_{n,i} = \frac{\partial L}{\partial \dot{r}_{n,i}} \tag{1.4}$$

In computing these partial derivatives we mean to vary L with respect to $r_{n,i}$, $\dot{r}_{n,i}$, and its explicit dependence on t, not its numerical value as a function of t. The latter is the variation dL/dt.

With these definitions Newton's third law

$$\mathbf{F}_n = \frac{d\mathbf{p}_n}{dt} = m_n \ddot{\mathbf{r}}_n \tag{1.5}$$

can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{n,i}} = \frac{\partial L}{\partial r_{n,i}} \tag{1.6}$$

or more concisely

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_k} = \frac{\partial L}{\partial r_k} \tag{1.7}$$

The index k now runs over all n and i. Equations (1.7) are Lagrange's equations.

1.1.2 Hamilton's Principle

Lagrange's equations make it easy to write the laws of motions in coordinates other than Cartesian coordinates, like spherical coordinates or confocal hyperbolic coordinates. One way to do that is to start in Cartesian coordinates and use the chain rule. The **action** provides a more elegant solution.

In general the Lagrangian can be a function of the time t as well as the coordinates and the velocities. Define the action as a functional of the coordinates and velocities:[1]

$$S = \int_{t_1}^{t_2} L(r_1 \ldots r_k, \dot{r}_1 \ldots \dot{r}_k, t)\, dt = \int_{t_1}^{t_2} L(r, \dot{r}, t)\, dt \tag{1.8}$$

In the second form the symbols r and $\dot{r}$ stand for $3n$ functions each. The endpoints of the integral are arbitrary fixed times. The value of the action depends on how the system gets from the configuration at t_1 to the configuration at t_2, subject only to the condition that it has the same coordinates at the start and at the finish. For any such path described by $r(t)$ the action has some value. Only one of these paths is a solution to Newton's law, and that one is an extremum of the action. To prove this, expand S about the correct path, keeping the values at the endpoints fixed. To first order,

$$\begin{aligned} S_o + \delta S &= \int_{t_1}^{t_2} L(r + \delta r, \dot{r} + \delta \dot{r}, t)\, dt \\ &= S_o + \sum_k \int_{t_1}^{t_2} \frac{\partial L}{\partial r_k} \delta r_k dt + \sum_k \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{r}_k} \delta \dot{r}_k dt + \cdots \end{aligned} \tag{1.9}$$

[1] A functional is a special kind of function, one whose argument is a function but whose value is a number.

But $\delta\dot{r}_k(t) = d\delta r_k/dt$, so

$$\begin{aligned}\delta S &= \sum_k \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial r_k}\delta r_k + \frac{\partial L}{\partial \dot{r}_k}\frac{d}{dt}\delta r_k \right) dt \\ &= \sum_k \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial r_k}\delta r_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_k}\delta r_k\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_k}\right)\delta r_k \right) dt\end{aligned} \tag{1.10}$$

The middle term is zero because $\delta\mathbf{r}_k = 0$ at the endpoints. So

$$\delta S = \sum_k \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial r_k}\delta r_k - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_k}\right)\delta r_k \right) dt = 0 \tag{1.11}$$

Newtonian mechanics is equivalent to the statement that the classical physical path is the one that minimizes (more precisely extremizes) the action.

Equation (1.11) remains true under a reparametrization of the coordinates and velocities. Let q_k be any $3n$ functions of the r_k, and write L as a function of q_k and $\dot{q}_k$:

$$0 = \delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \tag{1.12}$$

where q and $\dot{q}$ stand for the new collection of $3n$ coordinates and velocities. Equation (1.12) is **Hamilton's principle**. It has the form of a standard problem in the calculus of variations, like the brachistochrone problem first solved by Newton. The same steps as above, in the reverse order, lead to the **Euler-Lagrange equations** in any coordinates:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}} \tag{1.13}$$

Of course equation (1.13) can also be derived from Newton's law by manipulating the chain rule for partial derivatives, or from d'Alembert's principle.

Hamilton's principle makes it easy to describe systems where the force depends on the velocity, not just the position, of the particles. For velocity-independent potentials as above, equation (1.11) restates Newton's second law. While not all dynamical systems are so simple that they can be written like this in terms of a potential function, *all physical systems we know of can be written in terms of an action functional and a Lagrangian.*

Example: Charged particle in an electromagnetic field

The most important example that is not trivial is a particle moving in a prescribed electromagnetic field. In terms of the electrostatic potential $\phi(\mathbf{r}, t)$ and the magnetic vector potential $\mathbf{A}(\mathbf{r}, t)$, the fields are[2]

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{1.14}$$

The correct Lagrangian for a single charged particle is

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi(\mathbf{r}, t) + \frac{q}{c}\mathbf{A}(\mathbf{r}, t) \cdot \dot{\mathbf{r}} \tag{1.15}$$

[2] I use Gaussian (cgs) units throughout.

where q is the particle's electric charge. Here is the demonstration:

$$\begin{aligned}\frac{\partial L}{\partial r_i} &= -q\frac{\partial \phi}{\partial r_i} + \frac{q}{c}\sum_j \dot{r}_j \frac{\partial A_j}{\partial r_i} \\ \frac{\partial L}{\partial \dot{r}_i} &= m\ddot{r}_i + \frac{q}{c}A_i \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} &= m\ddot{r}_i + \frac{q}{c}\sum_j \frac{\partial A_i}{\partial r_j}\dot{r}_j + \frac{q}{c}\frac{\partial A_i}{\partial t}\end{aligned} \tag{1.16}$$

The Euler-Lagrange equations are

$$m\ddot{r}_i = -q\frac{\partial \phi}{\partial r_i} + \frac{q}{c}\sum_j \left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot{r}_j - \frac{q}{c}\frac{\partial A_i}{\partial t} \tag{1.17}$$

The middle two terms on the right can be written[3]

$$\frac{q}{c}\sum_j \left(\frac{\partial A_j}{\partial r_i}\dot{r}_j - \frac{\partial A_i}{\partial r_j}\dot{r}_j\right) = \frac{q}{c}\sum_{j,k}\epsilon_{kij}\dot{r}_j(\boldsymbol{\nabla}\times\mathbf{A})_k = \frac{q}{c}(\mathbf{v}\times\mathbf{B})_i \tag{1.18}$$

while the remaining two terms are

$$-q\left(\frac{\partial \phi}{\partial r_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right) = qE_i \tag{1.19}$$

Hamilton's principle for this Lagrangian is solved by

$$m\ddot{\mathbf{r}} = q\left(\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right) \tag{1.20}$$

Equation (1.20) is the Lorentz force law.

1.1.3 Canonical Momenta and the Hamiltonian Formulation

The momentum canonically conjugate to q_k is defined by

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{1.21}$$

For a Lagrangian of the form $L = T - V(r)$, the canonical momentum is the same as the mechanical momentum, but not in general. For the electromagnetic example above, the canonical momenta are

$$p_k = \frac{\partial T}{\partial \dot{r}_k} + \frac{q}{c}A_k(\mathbf{r},t) \qquad \text{or} \qquad \mathbf{p} = m\mathbf{v} + \frac{q}{c}\mathbf{A} \tag{1.22}$$

It is useful to express Hamilton's principle in terms of the q's and the p's. To do that one makes a Legendre transformation analogous to those used in elementary

[3] ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol. See Appendix A.1.

thermodynamics. Think of L as a functional with p instead of $\dot{q}$ as one of the independent variables. Then define the Hamiltonian as

$$H(p,q,t) = \sum_k p_k \dot{q}_k - L(p,q,t) \tag{1.23}$$

Example: Particle in a potential
In the simple potential case,

$$H = m\sum_k p_k \dot{q}_k - L = m\sum_k \left(\frac{p_k}{m}\right)^2 - L = 2T - T + V = T + V \tag{1.24}$$

The value of H will almost always be the energy, as it is here, but it is the functional dependence of H on the p's and the q's that is important.

Example: Charged particle in an electromagnetic field
For a particle in an electromagnetic field, the *value* of the Hamiltonian is indeed the energy:

$$\begin{aligned} H &= \mathbf{p}\cdot\dot{\mathbf{r}} - L = \left(m\mathbf{v} + \frac{q}{c}\mathbf{A}\right)\cdot\dot{\mathbf{r}} - \frac{1}{2}m\mathbf{v}^2 + q\phi - \frac{q}{c}\mathbf{A}\cdot\mathbf{v} \\ &= \frac{1}{2}mv^2 + q\phi \end{aligned} \tag{1.25}$$

The magnetic field $\mathbf{B}$ does not contribute to the value of the energy, but it does enter the equations of motion. In terms of $\mathbf{p}$ and $\mathbf{r}$ the Hamiltonian is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi \tag{1.26}$$

Hamilton's Equations
Expand the Hamiltonian for arbitrary q_k and p_k about the physical values:

$$\begin{aligned} \delta H &= \sum_k \left(\dot{q}_k \delta p_k + p_k \delta\dot{q}_k - \frac{\partial L}{\partial q_k}\delta q_k - \frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k\right) \\ &= \sum_k \left(\dot{q}_k \delta p_k - \frac{\partial L}{\partial q_k}\delta q_k\right) \end{aligned} \tag{1.27}$$

If the p_k and q_k are solutions to the Euler-Lagrange equations (1.13), then

$$\delta H = \sum_k \left(\dot{q}_k \delta p_k - \dot{p}_k \delta q_k\right) \tag{1.28}$$

Since the p_k and q_k are varied independently,

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \text{and} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}} \tag{1.29}$$

These are Hamilton's equations of motion. They have exactly the same content as equations (1.13).

Example: Particle in a potential

For a mechanical particle in Cartesian coordinates, $H = T + V$, and equations (1.29) are

$$\dot{r}_k = \frac{1}{m} p_k \qquad \text{and} \qquad \dot{p}_k = -\frac{\partial V}{\partial r_k} \tag{1.30}$$

Differentiate one more

$$\ddot{r}_k = \frac{1}{m}\dot{p}_k = -\frac{1}{m}\frac{\partial V}{\partial r_k} \tag{1.31}$$

or $m\ddot{\mathbf{r}} = \mathbf{F}$.

Example: Charged particle in an electromagnetic field

For a particle in an electromagnetic field, the equations of motion are

$$\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial r_k} = \frac{1}{m}\sum_i \left(p_i - \frac{q}{c}A_i\right)\frac{q}{c}\frac{\partial A_i}{\partial r_k} - q\frac{\partial \phi}{\partial r_k} \\ \dot{r}_k &= \frac{\partial H}{\partial p_k} = \frac{1}{m}\left(p_k - \frac{q}{c}A_k\right) \end{aligned} \tag{1.32}$$

Differentiate:

$$\begin{aligned} m\ddot{r}_k = \left(\dot{p}_k - \frac{q}{c}\dot{A}_k\right) &= q\sum_i \frac{v_i}{c}\left(\frac{\partial A_i}{\partial r_k} - \frac{\partial A_k}{\partial r_i}\right) - q\left(\frac{\partial \phi}{\partial r_k} + \frac{1}{c}\frac{\partial A_k}{\partial t}\right) \\ &= q\left(E_k + (\frac{\mathbf{v}}{c} \times \mathbf{B})_k\right) \end{aligned} \tag{1.33}$$

Again, one gets the Lorentz force law in the nonrelativistic limit.

1.2 CLASSICAL SPACE-TIME SYMMETRIES

Ever since Galileo discovered the principle of relativity, the invariance of the laws of nature under certain changes of the coordinate system has been a fundamental tenet of physics. That the laws are the same even if you rotate or translate the coordinates is the mathematical statement of the principle. No experiment can discover the origin of coordinates or the orientation of the coordinate axes; they are arbitrary.

In both classical and quantum mechanics, **symmetries** are closely connected to conservation laws. Why is that? One way to solve a problem in classical mechanics is to find coordinates and momenta that make the Hamiltonian H independent of some of the coordinates. Then the conjugate momenta will be constants of the motion. For instance if H is independent of the differences of the positions of the particles, the total momentum is conserved. If H is independent of the angle of a rotation in some plane, that component of angular momentum is conserved.

1.2.1 The Space-Time Transformations

For isolated physical systems, these transformations are all symmetries in the sense that when the coordinates undergo these transformations, for isolated systems, the form of the physical laws is unchanged.

Translations shift the coordinates by a vector **a**. There is no absolute origin to the coordinate system, and no experiment can find out where it is. Time translations change the value of a coordinate to its value at a different time. There is no absolute meaning to the setting of the clock or the calendar, and so no experiment can find out when $t = 0$ was or will be.

Rotations by an angle θ about the direction of a unit vector $\hat{\boldsymbol{n}}$ are described by a 3×3 real orthogonal matrix $\bar{R}$, one whose transpose is its inverse. Usually the word "rotation" excludes reflections, and so is reserved for real orthogonal 3×3 matrices that are *unimodular*, that is, have determinant $+1$. There is no absolute preferred orientation of the coordinate system. Boosts change the coordinates to ones moving with a uniform velocity relative to the first, and neither of the two frames of reference is preferred by nature. This is the principle of relativity that Galileo discovered.

> Note: The nonrelativistic form of the boosts given here is not really a symmetry of nature. In the real world Galilean boosts are replaced by the relativistic (Lorentz) form. We will study those in detail in Chapter 12.

Reflections change the sign of all three coordinates, and time inversion reverses the direction of the clocks. Reflection invariance means there is no absolute meaning to the distinction between left-handed coordinate systems and right-handed ones. Time-inversion invariance means that if you recorded the time of all events by a clock that is running backward, the laws of nature would be satisfied as well.

All the fundamental physical laws are invariant under translations, time translations, rotations, and Lorentz boosts. The most common laws are invariant under all the transformations listed in Table 1.1, but the weak interactions are not invariant under $\mathcal{R}$, and (probably) not even $\mathcal{T}$. The discrete symmetries are valid for electromagnetic, nuclear, and even gravitational forces.

Quantum systems have these symmetries and more: The additional ones are often called *internal* symmetries because they have nothing obvious to do with space and time. Examples of internal symmetries are charge conjugation or matter-antimatter symmetry, isotopic spin, and electromagnetic gauge invariance.

Table 1.1 shows what some symmetry transformations do to the Cartesian coordinates x, y, and z of a particle.[4]

Observables are usually functions of the momenta as well as the coordinates, so you have to know how the momenta transform too. It is not enough to know how the coordinates transform under a symmetry. In Cartesian coordinates the momenta transform as in Table 1.2.

The form of the equations of motion is unchanged under the transformations indicated, provided they are indeed symmetry transformations. Except for those that involve the time explicitly—time translation, time inversion, and the boosts—this means that the Hamiltonian must be invariant under the transformation.

What properties of a transformation are independent of the quantity being transformed? These are the universal structure of the transformations themselves,

[4]Rather than change coordinate systems as one does in general relativity, I will always define symmetries of a system by what a transformation does to observables. This is the "active" point of view. The opposite convention just replaces all the transformations with their inverses.

Translations:	$T(\mathbf{a})$	$\mathbf{r}_i \to \mathbf{r}_i + \mathbf{a}$
Time translations:	$U(t_o)$	$\mathbf{r}_i(t) \to \mathbf{r}_i(t + t_o)$
Rotations:	$R(\hat{\mathbf{n}}, \theta)$	$r_{i,m} \to \sum_{n=1}^{3} \bar{R}_{mn}(\hat{\mathbf{n}}, \theta) r_{i,n}$
Galilean boosts:	$G(\mathbf{v})$	$\mathbf{r}_i \to \mathbf{r}_i + \mathbf{v}t$
Reflections:	$\mathcal{R}$	$\mathbf{r}_i \to -\mathbf{r}_i$
Time inversion:	$\mathcal{T}$	$\mathbf{r}_i(t) \to \mathbf{r}_i(-t)$

TABLE 1.1: The space-time symmetries of classical physics: Coordinate transformations

Translations:	$T(\mathbf{a})$	$\mathbf{p}_i \to \mathbf{p}_i$
Time translations:	$U(t_o)$	$\mathbf{p}_i(t) \to \mathbf{p}_i(t + t_o)$
Rotations:	$R(\hat{\mathbf{n}}, \theta)$	$p_{i,m} \to \sum_{n=1}^{3} \bar{R}_{mn}(\hat{\mathbf{n}}, \theta) p_{i,n}$
Galilean boosts:	$G(\mathbf{v})$	$\mathbf{p}_i \to \mathbf{p}_i + m\mathbf{v}$
Reflections:	$\mathcal{R}$	$\mathbf{p}_i \to -\mathbf{p}_i$
Time inversion:	$\mathcal{T}$	$\mathbf{p}_i \to -\mathbf{p}_i(-t)$

TABLE 1.2: The space-time symmetries of classical physics: Momentum transformations

the rules for composing them. Under composition, the transformations form a **group**.[5] The composition rules, or group multiplication, are geometrical and can be taken over directly into quantum mechanics.

1.2.2 Translations

Now I want to examine translations and rotations in more detail. Translations are the simplest, since they all commute with one another:

$$T(\mathbf{a}_1)T(\mathbf{a}_2) = T(\mathbf{a}_1 + \mathbf{a}_2) = T(\mathbf{a}_2 + \mathbf{a}_1) = T(\mathbf{a}_2)T(\mathbf{a}_1) \tag{1.34}$$

You can verify that this rule holds for translations on $\mathbf{r}$ and $\mathbf{p}$ given above.

The properties of any continuous group of transformations are almost completely determined by the properties of very small transformations. For translations, this means that if you expand $T(\mathbf{a})$ in a power series in $\mathbf{a}$, you only have to learn about the first-order term in the expansion. Geometrically, the large translations are determined by the small ones because any $\mathbf{a}$ can in principle be found by composing, or integrating, a large number of small translations.

Let A be any observable. Under a small transformation $T(\boldsymbol{\epsilon})$, $\delta\mathbf{r}_k = \boldsymbol{\epsilon}$ and

[5] The definition of a group can be found in Section 4.1.

$\delta \mathbf{p}_k = 0$. Then $A \to A + \delta A$ (plus higher order terms) where

$$\delta A = \sum_n \frac{\partial A}{\partial r_n}\delta r_n + \sum_n \frac{\partial A}{\partial p_n}\delta p_n = \sum_{k,m} \frac{\partial A}{\partial r_{k,m}}\epsilon_m = \boldsymbol{\epsilon} \cdot \sum_k \boldsymbol{\nabla}_k A \tag{1.35}$$

1.2.3 Rotations

Rotations do not all commute with one another, so they are both more complicated and more interesting. A theorem (due to Euler) says that any rotation leaves some axis unchanged; so any rotation can be described by a unit vector $\hat{\boldsymbol{n}}$ and an angle θ of rotation about $\hat{\boldsymbol{n}}$. That is, any of the 3×3 matrices $\bar{R}$ has an eigenvector $\hat{\boldsymbol{n}}$ with eigenvalue unity: $\bar{R}\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}$. I have been using this result in the notation for the rotation matrices $\bar{R}(\hat{\boldsymbol{n}}, \theta)$.

The rotation matrices are worth learning about in some detail: Rotations about the z-axis have the form[6].

$$\bar{R}(\hat{\boldsymbol{n}}_z, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.36}$$

Approximate the matrix in equation (1.36) to first order in the angle:

$$\bar{R}(\hat{\boldsymbol{n}}_z, \epsilon) = 1 + \begin{pmatrix} 0 & -\epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.37}$$

The transformed coordinate is

$$x' = x - \epsilon y \qquad y' = y + \epsilon x \qquad z' = z \tag{1.38}$$

or

$$\mathbf{r}' = \mathbf{r} + \delta\mathbf{r} \tag{1.39}$$

where

$$\delta\mathbf{r} = \epsilon\hat{\boldsymbol{n}}_z \times \mathbf{r} + \cdots \tag{1.40}$$

There is nothing special about the z-axis. The result is general:

$$\mathbf{r}' = \mathbf{r} + \epsilon\hat{\boldsymbol{n}} \times \mathbf{r} + \cdots \tag{1.41}$$

or in components

$$\delta r_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j r_k + \cdots \tag{1.42}$$

Similarly, the first order change in the momentum is

$$\delta p_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j p_k + \cdots \tag{1.43}$$

[6] I reserve the notation $\bar{R}(\theta, \hat{\boldsymbol{n}})$ for these real 3×3 matrices, to distinguish them from the abstract operation $R(\theta, \hat{\boldsymbol{n}})$ that appears in Tables 1.1 and 1.2. This is not a standard notation, but I find it very convenient.

First Order Transformations of Vectors

Examples of vectors observables are $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$. Any three observables V_i components of a **vector** provided that under a rotation,

$$V_i \to \sum_j \bar{R}_{ij} V_j \tag{1.44}$$

Then under a small rotation,

$$\delta V_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j V_k \tag{1.45}$$

or $\delta \mathbf{V} = \epsilon \hat{\boldsymbol{n}} \times \mathbf{V}$. Observables that like the energy are invariant under rotations, are called **scalars**.

Vector and Scalar Observables

In classical mechanics all observables are functions of the r_k and the p_k, so the transformation properties of other observables can be deduced from the rules for $\mathbf{r}$ and $\mathbf{p}$. For example, the angular momentum of a particle is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Under rotations (exercise!)

$$L_m \to \sum_{n=1}^{3} \bar{R}(\hat{\boldsymbol{n}}, \theta)_{mn} L_n \tag{1.46}$$

Fields

Fields—like the electric or the gravitational field—are observables that are functions of space and time. They also transform in definite ways. The electrostatic potential $\phi(\mathbf{r}, t)$ introduced in equation (1.14) is an example of a scalar field. If you rotate a field, the value at some point $\mathbf{r}$ after rotation is the same value it had before rotation at the point that rotates into $\mathbf{r}$. That is,

$$\phi(\mathbf{r}, t) \to \phi(\mathbf{r}'', t) \qquad \text{where} \qquad r_i'' = \sum_j \bar{R}_{ij}^{-1} r_j \tag{1.47}$$

Another way to write this rule is $\phi(\mathbf{r}, t) \to \phi'(\mathbf{r}, t)$, where

$$r_i' = \sum_j \bar{R}_{ij} r_j \qquad \text{and} \qquad \phi'(\mathbf{r}', t) = \phi(\mathbf{r}, t) \tag{1.48}$$

The transformation rule for fields like the magnetic vector potential is $A_i(\mathbf{r}, t) \to A_i'(\mathbf{r}, t)$, where

$$r_i' = \sum_j \bar{R}_{ij} r_j \qquad \text{and} \qquad A_i'(\mathbf{r}', t) = \sum_j \bar{R}_{ij} A_j(\mathbf{r}, t) \tag{1.49}$$

Reflections and Rotations

Ordinary vectors, like $\mathbf{r}$ and $\mathbf{p}$, are odd under reflections. Ordinary scalars, like $\mathbf{p}^2/2m$, are even under reflections. Vectors that have the opposite sign from $\mathbf{r}$ and $\mathbf{p}$ under reflections, or scalars that have the opposite sign from the energy under reflections, are called **pseudovectors** and **pseudoscalars**. Under a reflection $\mathbf{L} \to$