On the fundamental group of compact manifolds of non-positive curvature

By Shing Tung Yau

1. Introduction

Let M be a compact, C^∞ riemannian manifold of non-positive curvature with fundamental group $\pi_1(M)$. It is well known that M is a $K(\pi_1(M), 1)$ and therefore completely determined up to homotopy type by $\pi_1(M)$. Furthermore, every element of $\pi_1(M)$ must have infinite order, and in the special case that M has strictly negative curvature, every abelian subgroup of $\pi_1(M)$ must be cyclic. The natural questions then arise whether deeper statements can be made regarding the structure of the group $\pi_1(M)$ and to what extent the structure of the group $\pi_1(M)$ influences the riemannian structure of M.

The purpose of this paper is to establish the following facts in this regard.

THEOREM 1. *Every solvable subgroup of $\pi_1(M)$ is of finite index over a finitely generated abelian group. In fact, every solvable subgroup of $\pi_1(M)$ is a so-called Bieberbach group.*

THEOREM 2. *Let G be any subgroup of $\pi_1(M)$. If A is a subgroup which is subnormal and maximal abelian in G, then G is of finite index over A and hence a Bieberbach group.*

Theorem 1 gives in the following corollary an affirmative answer to a question raised by J. A. Wolf [4].

COROLLARY 1. *If $\pi_1(M)$ is solvable, then M is a flat manifold.*

There are also the following consequences of Theorems 1 and 2.

COROLLARY 2. *Every solvable subgroup of $\pi_1(M)$ is finitely generated.*

The following result has also been obtained by William Byers (*Generalization Of A Theorem Of Preissmann*, Proc. Amer. Math. Soc.).

COROLLARY 3. *If M has strictly negative curvature, then every subgroup of $\pi_1(M)$ which contains a subnormal abelian subgroup is cyclic, and hence every solvable subgroup is cyclic.*

The author wishes to express his deep gratitude to Professor H. B. Lawson for his many invaluable suggestions. He also thanks Professors

Chern, Kobayashi and Sah for several helpful discussions.

Note. The author has subsequently learned that a proof of the first part of Theorem 1 has also been recently given by J. A. Wolf and D. Gromoll.

2. Basic lemmas

Throughout this part, we shall assume our manifold M is compact and of non-positive curvature. The proof of the Flat Torus Theorem in [3, § 2] essentially shows the following:

LEMMA 1. *Let $\langle\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\rangle$ be a finitely generated abelian subgroup of $\pi_1(M, p)$ for some $p \in M$. Then there exists a euclidean space R^k totally geodesically embedded in the universal cover $\tilde{M}$ of M such that R^k is invariant under the deck transformations corresponding to $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$.*

LEMMA 2. *Every abelian subgroup of $\pi_1(M)$ is finitely generated.*

Proof. We shall consider our group $\pi_1(M)$ as a group of deck transformations acting on $\tilde{M}$, the universal cover of M. Since it acts freely on $\tilde{M}$, it is immediate from Lemma 1 that every finitely generated abelian subgroup of $\pi_1(M)$ has rank $\leqq \dim M$. Let A be any abelian subgroup of $\pi_1(M)$. Let $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ be an abelian subgroup of maximal rank $(= n)$ in A. By Lemma 1, there exists a totally geodesic euclidean space R^n embedded in $\tilde{M}$ which is invariant under $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$. From now on we shall denote such a euclidean space by $R(\alpha_1, \alpha_2, \cdots, \alpha_n; o)$ where o is an arbitrary fixed point in this euclidean space. We shall also consider $R(\alpha_1, \alpha_2, \cdots, \alpha_n; o)$ as a vector space with origin o. The transformations $\alpha_1, \alpha_2, \cdots, \alpha_n$ corresponding to translations $t_{\tilde{\alpha}_1}, t_{\tilde{\alpha}_2}, \cdots, t_{\tilde{\alpha}_n}$ where $\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n$ are the displacement vectors corresponding to $\alpha_1, \alpha_2, \cdots, \alpha_n$ respectively. Let $C = \{x \mid x = \sum_{i=1}^n a_i\tilde{\alpha}_i \in R(\alpha_1, \alpha_2, \cdots, \alpha_n; o),\ 0 \leqq a_i \leqq 1\}$. Then obviously C is a bounded parallelopiped in $R(\alpha_1, \alpha_2, \cdots, \alpha_n; o)$. Suppose $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ is not equal to A. Then for all $\beta \in A \sim \langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ there exists $k > 0$, such that $\beta^k \in \langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$, because $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ is of maximal rank. Let β_1 be such an element. Obviously, we may assume that there exist $i_1, i_2, \cdots, i_n$, with $0 \leqq i_j < k$ for $j = 1, \cdots, n$ such that $\beta_1^k = \alpha_1^{i_1}\alpha_2^{i_2} \cdots \alpha_n^{i_n}$. By Lemma 1, there exists an $R(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1; o_1)$ for some o_1 in M. (Note that in applying the Flat Torus Theorem of [3, § 2], one may think that the base point of $\tilde{M}$ is changed when we minimize the sum of the lengths of the geodesic loops. In that case, $\langle\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1\rangle$ are changed to $\langle g^{-1}\alpha_1 g, g^{-1}\alpha_2 g, \cdots, g^{-1}\alpha_n g, g^{-1}\beta_1 g\rangle$ for some $g \in \pi_1(M, p)$. However, the difficulty is overcome by noting that $gR(g^{-1}\alpha_1 g, g^{-1}\alpha_2 g, \cdots, g^{-1}\alpha_n g, g^{-1}\beta_1 g; o)$ is invariant under $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ where o is the original base point.) Let

C_1 be the corresponding parallelopiped for $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ in $R(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1; o_1)$. Then since the infinite geodesics which extend the edges of C and C_1 are translated by $\alpha_1, \alpha_2, \cdots, \alpha_n$ respectively, the corresponding geodesics are of the same type. Hence, the arguments of [3] show that the corresponding edges of C and C_1 are of equal length. The diameters of C and C_1 are therefore bounded by a constant depending only on $\alpha_1, \alpha_2, \cdots, \alpha_n$. Now from the relation $\beta_1^k = \alpha_1^{i_1}\alpha_2^{i_2} \cdots \alpha_n^{i_n}$, we know that if $\tilde{\beta}_1$ is the displacement vector corresponding to β_1, then $\tilde{\beta}_1 \in C_1$ and of course $\tilde{\beta}_1 \neq \tilde{\alpha}_j$ for $j = 1, 2, \cdots, n$. Thus, if A is not finitely generated, we can construct inductively a sequence of parallelopipeds C_i with vertices o_i such that for each i, C_i contains the $n + i$ distinct points $\alpha_1 o_i, \cdots, \alpha_n o_i, \beta_1 o_i, \cdots, \beta_i o_i$. Since M is compact, we may assume there exists a sequence $\{g_i\} \subset \pi_1(M)$ such that $g_i(o_i) \to p$ for some $p \in M$. Let L be the uniform bound for the diameters of the C_i's. Then L also bounds the diameter of each $g_i C_i$. Now consider the ball B of radius $2L$ at p. For every sufficiently large i, B contains $n + i$ distinct images of a point under the deck group of the covering. This contradicts the proper discontinuity of this group and establishes the lemma.

LEMMA 3. *Let G be a subgroup of $\pi_1(M)$. If G is normal over a maximal abelian subgroup $A = \langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$, then G is of finite index over A.*

Proof. Assume without loss of generality that A has rank n. As above let $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n$ be the respective displacement vectors $\overrightarrow{o\alpha_1(o)}, \cdots, \overrightarrow{o\alpha_n(o)}$ in the euclidean space $R(\alpha_1, \cdots, \alpha_n; o)$. Then for each i, $g^{-1}\alpha_i g \in \langle \alpha_1, \cdots, \alpha_n \rangle$ and thus, $g^{-1}\alpha_i g = \sum_{j=1}^n n_{ij}\alpha_j$ for integers n_{ij}. Consider the vector group $\tilde{A} = \langle \tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n \rangle$ with the induced metric from $R(\alpha_1, \alpha_2, \cdots, \alpha_n; o)$. Let $R_g \colon \tilde{A} \to \tilde{A}$ be the homomorphism defined by $R_g(\tilde{\alpha}_i) = \sum_{j=1}^n n_{ij}\tilde{\alpha}_j$ for $i = 1, \cdots, n$. We shall prove that R_g is an isometry.

In fact, since g is an isometry, it maps $R(\alpha_1, \alpha_2, \cdots, \alpha_n; o)$ to a totally geodesic submanifold M' which is again isometric to a flat euclidean space. This manifold M' is also invariant under $\alpha_1, \alpha_2, \cdots, \alpha_n$ due to the fact that A is normal in G. Hence, it is trivial that α_i translates the geodesic passing through $g(o)$ and $\alpha_i g(o)$. By the arguments of [3], we know that for all i, $d(g(o), \alpha_i g(o)) = d(o, \alpha_i(o))$, where d denotes the distance in M. However, for each i, $d(g(o), \alpha_i g(o)) = d(o, g^{-1}\alpha_i g(o))$, and thus $d(o, \alpha_i(o)) = d(o, g^{-1}\alpha_i g(o))$ which proves that R_g is an isometry of A.

The number of such isometries is finite, for if L is the maximum length of the vectors $\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n$, then the closed ball $B(L)$ contains $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n$ and finitely many other vectors $\tilde{\alpha}_{n+1}, \cdots, \tilde{\alpha}_m$ of A. Since all the R_g's are isometries, they must map $B(L)$ onto itself. Since the set of points $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n$,

$\tilde{\alpha}_{n+1}, \cdots, \tilde{\alpha}_m$ is left invariant and R_g is determined by its values on $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n$, the number of distinct R'_gs actually divides $m!$. We now note that $R_b = R_c$ if and only if $b^{-1}\alpha_i b = c^{-1}\alpha_i c$, for all i. (Recall that G acts freely.) This means $R_b = R_c$ if and only if $cb^{-1}\alpha_i bc^{-1} = \alpha_i$ for all i. By the maximality of A, we know that $R_b = R_c$ if and only if $cb^{-1} \in A$. Hence, A has only a finite number of cosets in G, i.e. $[G: A] < \infty$.

Remark: G is a so called Bieberbach group.

LEMMA 4. *Let $1 \lhd F \lhd G$ be groups. Suppose F is finite and G/F is abelian. Then there exist subgroups A_1, A_2 of G such that: $1 \lhd A_1 \lhd A_2 \lhd G$, A_1 is a finite group contained in the center of A_2, A_2/A_1 is abelian, and G/A_2 is finite.*

Proof. Let A_2 be the centralizer of F in G. Then $A_2 \lhd G$, and G/A_2 is finite. In fact, consider the action of G on F by inner automorphisms; then the kernel of such an action is A_2 and G/A_2 is isomorphic to a subgroup of $\text{Aut}(F)$ which is finite. Now let $A_1 = A_2 \cap F$. Then obviously A_1 is finite abelian and is contained in the center of A_2. Finally, $A_2/A_1 = A_2/A_2 \cap F \cong A_2 \cdot F/F$ is abelian.

LEMMA 5. *Let $1 \lhd A_1 \lhd A_2$ be as in Lemma 4. Then there exists a subgroup A'_1 where $1 \lhd A'_1 \lhd A_2$ and A'_1 is abelian, and there exists an integer n such that for all $a \in A_2$, $a^n \in A'_1$.*

Proof. Consider the group $A_2 = \langle a^n \mid a \in A_2 \rangle$ where n is the cardinal of A_1. We claim that A_2 is abelian. In fact, for all $a, b \in A_2$, $ab = bat$ for some $t \in A_1$. Hence, $a^2b = abat = batat = ba^2t^2$ (since t is in the center of A_2) and thus $a^nb = ba^nt^n = ba^n$.

3. The main theorems

THEOREM 1. *Let M be a compact riemannian manifold of non-positive curvature. Let $G < \pi_1(M)$ be solvable. Then there exists an abelian subgroup $A < G$ such that $[G: A] < \infty$, and G is in fact a Bieberbach group.*

Before proceeding to the proof, we shall examine some consequences of this theorem.

COROLLARY 1. (Wolf's conjecture.) *Let M be a compact riemannian manifold of non-positive curvature. If $\pi_1(M)$ is solvable, then M is flat.*

Proof. By Theorem 1, there exists an abelian subgroup A in $\pi_1(M)$ with $[\pi_1(M): A] < \infty$. Let $\tilde{M}$ be a cover of M with fundamental group $= A$. Then $\tilde{M}$ is compact and hence by lemma 1, $\tilde{M}$ is flat. Therefore M is flat.

COROLLARY 2. *Let M be a compact manifold of non-positive curvature. Then every solvable subgroup of M is finitely generated.*

Proof of Theorem 1. Let $1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$ be a subnormal series such that G_{i+1}/G_i is abelian for $i = 0, 1, 2, \cdots, n-1$. Let G_1' be a maximal abelian subgroup of G_2 which contains G_1. Then since G_2/G_1 is abelian, G_1' is normal in G_2. Lemma 3 shows that $[G_2: G_1'] = k < \infty$. Consider now the partial series $1 = G_0 \lhd G_1' \lhd G_2 \lhd G_3$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ be a basis of G_1', and let a be an arbitrary element of G_3. Then for each i, $a^{-1}\alpha_i^k a \in \langle \alpha_1, \alpha_2, \cdots, \alpha_p \rangle$, since G_2/G_1' is finite of order k. Let G_3' be the centralizer subgroup of $\langle \alpha_1^k, \alpha_2^k, \cdots, \alpha_p^k \rangle$ in G_3. We claim that $[G_3: G_3'] < \infty$. The proof is similar to that of Lemma 3. Namely, for each $a \in G_3$, consider the mapping $R_a: \langle \tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_p \rangle \to \langle \tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_p \rangle$ defined by $R_a(\tilde{\alpha}_i) = \widetilde{a^{-1}\alpha_i^k a}$ for $i = 1, 2, \cdots, p$. (The $\tilde{\alpha}_i$'s are as defined in Lemma 3.)

This map stretches the length of every vector by the factor k. In fact, the isometry a maps $R(\alpha_1, \alpha_2, \cdots, \alpha_p; o)$ to M', a totally geodesic submanifold isometric to $\mathbf{R}^p$. Since for each $ay \in M'$, we have $\alpha_i^k ay = a\alpha' y \in M'$ for some $\alpha' \in \langle \alpha_1, \alpha_2, \cdots, \alpha_p \rangle$, we know that M' is invariant under α_i^k for each i. Hence,

$$d(o, a^{-1}\alpha_i^k a(o)) = d(o, \alpha_i^k(o)) = kd(o, \alpha_i(o)) .$$

Let $B(L)$, $B(kL)$ be as defined in Lemma 3. Then every such R_a maps $B(L)$ into $B(kL)$. Since both $B(L)$ and $B(kL)$ contain a finite number of α's and since R_a is determined by its values on $\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_p$, the number of such R_a must be finite. As in Lemma 3, $R_a = R_b$ if and only if $ba^{-1}\alpha_i^k ab^{-1} = \alpha_i^k$ for all i, i.e. $R_a = R_b$ if and only if $ba^{-1} \in G_3'$. Hence $[G_3: G_3'] < \infty$. We now have the situation

$$1 = G_0 \lhd G_1'^k \lhd G_1' \lhd G_2 \cap G_3' \lhd G_3' < G_3$$

where $G_1'^k = \langle \alpha_1^k, \cdots, \alpha_p^k \rangle$ and $G_1'^k$ is normal in G_3'. Hence,

$$[G_2 \cap G_3': G_1'^k] \leqq [G_2: G_1'][G_1': G_1'^k] < \infty$$

and $G_3'/G_2 \cap G_3' \cong G_2 \cdot G_3'/G_2$ is abelian. By passing to the quotient by $G_1'^k$ and applying Lemmas 4 and 5, we know that there exist solvable subgroups G_2', G_2'' such that

$$1 \lhd G_1'^k \lhd G_2' \lhd G_2'' \lhd G_3' < G_3$$

with $G_2'/G_1'^k$ abelian and $[G_3': G_2''] < \infty$, and there exists an integer n such that for all $g \in G_2''$, $g^n \in G_2'$. By Lemma 3, we may enlarge $G_1'^k$ to a maximal abelian subgroup $G_1^* \lhd G_2'$ of finite index. Hence, there exists an m such that for all $g \in G_2''$, $g^m \in G_1^*$. We assert that if $A < G$ are solvable subgroups of $\pi_1(M)$, if A is abelian and if there exists an m such that for all $g \in G$, $g^m \in A$, then there exists an abelian subgroup $A' < G$ such that $[G: A'] < \infty$. In fact, let $A = \langle \alpha_1, \alpha_2, \cdots, \alpha_k \rangle$. Then as above if G' is the centralizer of $A^m = \langle \alpha_1^m, \cdots, \alpha_k^m \rangle$ in G, then $[G: G'] < \infty$. Thus, we are reduced to the case $A \lhd G$ where A is abelian, G centralizes A and where there exists an integer m, such that for all $g \in G$, $g^m \in A$. However, under these new conditions we claim that G is itself abelian. In fact, let $g_1, g_2 \in G$ be arbitrary. Consider the group G' generated by A, g_1, and g_2. This group is finitely generated and centralizes A. Furthermore, G'/A is finite. Then by a theorem of Schur-Baer [1], the commutator group $[G', G']$ is finite. However, since no element of $\pi_1(M)$ has finite order, we have $[G', G'] = 1$. Thus G' is abelian, and in particular $g_1 g_2 = g_2 g_1$. This proves the assertion. We now have the following situation:

$$1 \lhd G_1'' < G_2'' \lhd G_3' < G_3 \lhd G_4 \lhd \cdots$$

where G_1'' is constructed in our assertion (with $A = G_1^*$ and $G = G_2''$), and where

$$[G_3: G_1''] = [G_3: G_3'][G_3': G_2''][G_2'': G_1''] = q < \infty .$$

The technique above then shows that if G_4' is the centralizer of $G_1''^q$ in G_4, then $1 \lhd G_1''^q \lhd G_3 \cap G_4' \lhd G_4' < G_4$, with $[G_4: G_4'] < \infty$, $G_4'/G_3 \cap G_4'$ abelian, and $[G_3 \cap G_4': G_1''^q] < \infty$. In this way we climb up to the group $G_n = G$ and prove the first part of our theorem.

It remains to prove the last assertion of our theorem. Let A be an abelian subgroup of G such that $[G: A] < \infty$. Then by the counting principle of group theory, we know that the number of distinct conjugate subgroups of A in G is finite. (The number is actually equal to the index of the normalizer of A in G which is finite.) Hence the subgroup $A' = \bigcap_{g \in G} gAg^{-1}$ is normal and of finite index in G. Of course, A' is still abelian. We claim that the centralizer $Z(A')$ of A' in G is normal in G. In fact, let b be an arbitrary element in $Z(A')$. Then for all $a \in A'$, $g \in G$, we have, as A' is normal in G, $gag^{-1}b = bgag^{-1}$, i.e. $ag^{-1}bg = g^{-1}bga$. This implies that for all $g \in G$, $g^{-1}bg \in Z(A')$. Hence $Z(A')$ is normal in G. Since A' is of finite index in $Z(A')$, we know from our assertion above that $Z(A')$ is also abelian. We have constructed, therefore a sequence of abelian subgroups in G:

$$A' = Z^0(A') < Z^1(A') < Z^2(A') < \cdots < Z^n(A') < Z^{n+1}(A') < \cdots$$

where $Z^{n+1}(A')$ is the centralizer of $Z^n(A')$ in G, for all $n = 1, 2$. By Lemma 2, we know that every abelian subgroup is finitely generated. Hence, $\bigcup_{i=0}^{\infty} Z^i(A')$ is a finitely generated abelian subgroup, and thus for some n, $Z^n(A') = Z^{n+1}(A')$, i.e. $Z^n(A')$ is maximal abelian in G. The argument above then shows that $Z^n(A')$ is normal and of finite index in G. By definition, G is a Bieberbach group. This completes the proof of the theorem.

The same technique can be used to prove the following theorem.

THEOREM 2. *Let M be a compact riemannian manifold of non-positive curvature. Let $G < \pi_1(M)$ be an arbitrary subgroup. If A is a subgroup which is subnormal and maximal abelian in G, then G is of finite index over A and hence a Bieberbach group, i.e. if A is maximal abelian in G and if there exist subgroups $A_1, A_2, \cdots, A_{n-1} < G$ such that $A \lhd A_1 \lhd A_2 \lhd \cdots \lhd A_{n-1} \lhd G$ then $[G: A] < \infty$.*

COROLLARY 3. *If M is a compact manifold of strictly negative curvature, then every subgroup of $\pi_1(M)$ which contains a subnormal abelian subgroup is cyclic. In particular, every solvable subgroup of $\pi_1(M)$ is cyclic.*

Proof. Let G be the subgroup and A the abelian subgroup. Observe that by Lemmas 1 and 2 every abelian subgroup of $\pi_1(M)$ is cyclic in the strictly negative curvature case. Thus A is cyclic and following the arguments of Theorem 1 we see that A is of finite index over a cyclic group. Since G is a Bieberbach group, G must be cyclic.

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J. DIFFERENTIAL GEOMETRY
7 (1972) 211–228

COMPACT MANIFOLDS OF NONPOSITIVE CURVATURE

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0. Introduction and statement of results

Let M be a compact C^∞ riemannian manifold of nonpositive curvature[1] and with fundamental group π. It is well known [8, p. 102] that M is a $K(\pi, 1)$ and thus completely determined up to homotopy type by π. In light of this fact it is natural to ask to what extent the riemannian structure of M is determined by the structure of π, and the intent of this paper is to demonstrate that rather strong implications of this sort exist.

In the case that M has strictly negative curvature, the group π is known to be highly noncommutative. Every abelian, in fact, every solvable, subgroup of π is cyclic [3]. It is therefore a plausible conjecture that in the nonpositive curvature case, π will possess large amounts of commutativity only under special geometric circumstances. We shall show that this is true, that indeed those properties of π which involve commutativity have a dramatic reflection in the riemannian structure of M.

Our first theorem concerns abelian subgroups of π, which, since no element of π has finite order [8, p. 103], must be torsion free. As remarked above, when M is negatively curved, every abelian subgroup has rank one. However, when the curvature of M is simply nonpositive, we prove the following.

The flat torus theorem. *There exists an abelian subgroup of rank k in π if and only if there exists a flat k-torus isometrically and totally geodesically immersed in M.*

The second theorem concerns the case where π is a product of groups. In particular, we shall prove:

The splitting theorem. *Let M be real analytic and assume that π has no center. If π can be expressed as a direct product of groups $\pi = \mathscr{A}_1 \times \cdots \times \mathscr{A}_N$, then M is isometric to a riemannian product $M = M_1 \times \cdots \times M_N$, where $\pi_1(M_k) = \mathscr{A}_k$ for $k = 1, \cdots, N$.*

It is shown in § 4 that in the case that π has a nontrivial center, the splitting theorem, as stated, is not true. However, by a slight weakening of the conclusion, one can obtain a similar theorem for the general case.

As one may by now suspect, the appearance of a nontrivial center in π must

Received July 24, 1970 and, in revised form, July 21, 1971.
[1] Throughout the paper curvature refers to sectional curvature.

also have strong geometric consequences. In fact, from the work of J. A. Wolf in [12] one has

The center theorem. *Let $\mathscr{Z}$ be the center of π. Then $\mathscr{Z} \simeq k\boldsymbol{Z}$ for some $k \geq 0$, and there exists a foliation of M by totally geodesic, flat k-tori. Furthermore, there exists an abelian covering $T^k \times M' \to M$ of M by a riemannian product of a flat torus and another manifold M′. Let $\mathscr{N} = \pi_1(M')$ and let $\mathscr{A}$ be the abelian covering group. Then $\mathscr{N}$ is a normal subgroup of π which contains $[\pi, \pi]$, and the following sequences are exact:*

$$1 \to \mathscr{Z} \times \mathscr{N} \to \pi \to \mathscr{A} \to 1 ,$$
$$0 \to \mathscr{Z} \times (\mathscr{N}/[\pi, \pi]) \to H_1(M, \boldsymbol{Z}) \to \mathscr{A} \to 0 .$$

As particular consequences of this theorem we have that if $\mathscr{Z} \simeq k\boldsymbol{Z}$, then:

(a) there exist k linearly independent globally parallel vector fields on M,

(b) the torus group T^k acts effectively by isometries on M.

In § 5 we show that these geometric quantities completely characterize the center of π, namely:

(a′) *Suppose there exist exactly k linearly independent globally parallel vector fields on M. Then* rank $(\mathscr{Z}) = k$.

(b′) *Let I(M) be the group of isometries of M. Then $I(M)^0 \simeq T^k$ where $k =$* rank $(\mathscr{Z})$. *Furthermore, if $g \in I(M) \sim I(M)^0$, then g is not homotopic to the identity.*

Part (b′) together with the center theorem gives a generalization of a theorem of *T*. Frankel to manifolds of nonpositive curvature (§ 6).

In all of the above theorems the compactness of M is required. In fact, Bishop and O'Neill have shown that there exists a complete metric of constant negative curvature on $\boldsymbol{R} \times$ F where F is any compact manifold which admits a flat riemannian metric (e.g., a torus) [2, Cor. 7.10].

However, in the last section we show that certain of the above results can be shown to hold for complete nonpositively curved manifolds of finite volume. In particular, a form of Gottlieb's theorem is established for such cases.

We are indebted to *S*. Kobayashi for several helpful suggestions.

Note. Since writing this paper we have learned that J. A. Wolf and D. Gromoll[2] have obtained independent and somewhat different proofs of the first two theorems, including a C^∞ version of the splitting theorem.

1. Definitions, notation and basic lemmas

Thoughout the proofs of the main theorems of this paper we will need to make repeated use of certain established facts concerning manifolds of non-

[2] **Added in proof.** D. Gromoll & J. A. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature*, Bull. Amer. Math. Soc. **77** (1971) 545–552.