Part 3: Pre-Calculus

## 第3部分：

# 微积分初步

本部分内容为微积分的基础知识。微积分(calculus)产生于17世纪，是高等数学中研究函数的微分(differentiation)和积分(integration)，以及有关概念和应用的数学分支。微积分描述的是连续的变化，无论是现实世界的运动，还是纯抽象的代数曲线，这些变化都是无穷小的瞬时变化。微积分的发明堪称人类智慧的结晶，被称为“数学创造的最有效的科学研究工具”。

牛顿和莱布尼茨两位科学巨匠之间的“到底谁先发明了微积分”之争直到他们离世都没有得出分晓。现在我们称微积分公式为“牛顿-莱布尼茨公式”，不知道这对“冤家”看到后人把他们的名字写在一起会做何感想。

通过这一部分的学习，学生对函数的概念会更加熟悉，本部分内容包括但不局限于函数运算、因式分解、多项式除法、多项式函数、有理函数、倒数函数、反函数、对数函数，最后会介绍微积分的极限与导数。导数与上述的连续变化密切相关。此外，本部分也介绍了代数的一些重要课题，与数学竞赛、大学数学分析密不可分，包括但不局限于三维空间、矩阵、复数、圆锥曲线、级数、概率和高等三角函数。

# 22. Systems of Linear Inequalities with Two Variables 二元一次不等式组

Review Functional Inequalities in Section 7.6. 复习7.6节函数不等式。

## Linear Programming 线性规划

n. [ˈlɪniər ˈproʊ.ɡræm.ɪŋ]

Definition: The process of writing multiple linear inequalities related to some situation and finding the optimal value (minimum or maximum) of a linear objective function. 线性规划是指对一个环境列出若干线性不等式,并求出目标线性函数最值的步骤。

Each inequality is called a constraint. 每条不等式称为约束条件。

The region bounded by the graphs of the inequalities is called the feasible region. 被不等式图像围成的区域称为可行域。

The objective function is called the optimization equation. It must be linear. 目标函数必须为线性。

The maximum or minimum value of the optimization equation always occurs at one of the vertices of the feasible region. 目标函数的最值必然发生在可行域中的一个顶点上。

Examples: As a tutor in English and mathematics, Bob offers two types of services. He allots 40 minutes for an English lesson and 60 minutes for a math lesson. He cannot tutor more than 6 math lessons per day. Every day he has 10 hours available for lessons. If an English lesson costs \$60 and a math lesson costs \$75, what is a combination of numbers of English and math lessons that will maximize Bob's income per day?

Let $e$ be the number of English lessons and $m$ be the number of math lessons.

We have $\begin{cases} e \geqslant 0 \\ m \geqslant 0 \\ m \leqslant 6 \\ 40e + 60m \leqslant 600 \end{cases}$.

Our optimization equation is $I(e, m) = \$60e + \$75m$, from which we want to find its maximum value.

We graph the system of equations as shown in Figure 22-1.

We find the vertices $(e, m)$ to be $(0,0)$, $(0,6)$, $(6,6)$, and $(15,0)$.

We plug each vertex into our optimization equation, as shown in Table 22-1:

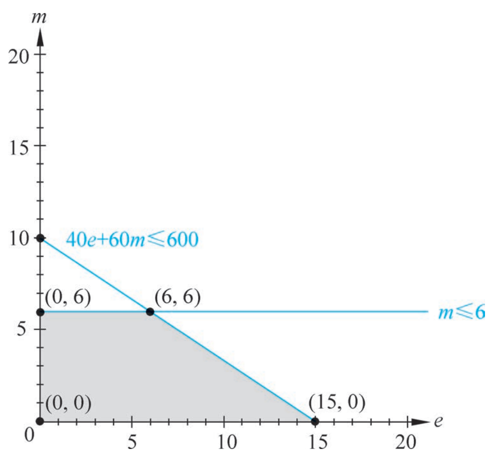


Figure 22-1

Table 22-1

| Input $(e,m)$ 输入 | Output $I(e,m)=\$60e+\$75m$ 输出 |
|---|---|
| (0,0) | \$ 0 |
| (0,6) | \$ 450 |
| (6,6) | \$ 810 |
| (15,0) | \$ 900 |

We know that (15, 0) produces the maximum output. In other words, tutoring 15 English lessons per day produces the maximum income.

# 23. The Three-Dimensional Space 三维空间

## 23.1 Introduction 介绍

Three-Dimensional Space 三维空间

n. [θri dəˈmenʃənəl speɪs]

Definition: The space determined by three mutually perpendicular axes: $x$-axis, $y$-axis, and $z$-axis. 三维空间是被三条互相垂直的轴（$x$ 轴、$y$ 轴、$z$ 轴）确定的空间。

Compared to the coordinate plane, the three-dimensional space has one extra dimension ($z$-axis), which represents the height of a plane. 与坐标平面比起来，三维空间多了一个维度（$z$ 轴），代表一个平面的高。

Many terminologies from the coordinate plane are analogues for those of the three-dimensional space.　坐标平面与三维空间的很多术语均类似。

The three-dimensional space is shown in Figure 23-1.

Properties：1. Each point on the three-dimensional plane can be specified by an ordered triple of numbers.　三维空间上的每个点都能用有序三元组表示。

2. (1) The $xy$-plane is a plane that contains the $x$-axis and $y$-axis and is perpendicular to the $z$-axis, as shown in Figure 23-2.

$xy$ 平面包含 $x$ 轴和 $y$ 轴，与 $z$ 轴垂直，如图 23-2 所示。

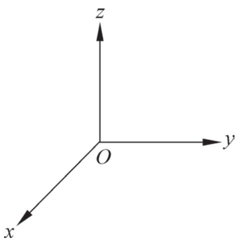


Figure 23-1

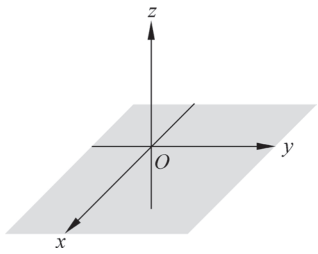


Figure 23-2

(2) The $xz$-plane is a plane that contains the $x$-axis and $z$-axis and is perpendicular to the $y$-axis, as shown in Figure 23-3.

$xz$ 平面包含 $x$ 轴和 $z$ 轴，与 $y$ 轴垂直，如图 23-3 所示。

(3) The $yz$-plane is a plane that contains the $y$-axis and $z$-axis and is perpendicular to the $x$-axis, as shown in Figure 23-4.

$yz$ 平面包含 $y$ 轴和 $z$ 轴，与 $x$ 轴垂直，如图 23-4 所示。

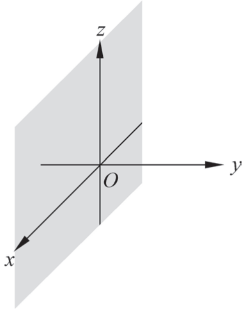


Figure 23-3

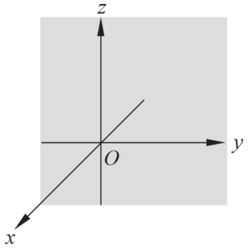


Figure 23-4

## Coordinate Axis　坐标轴

n. [koʊˈɔrdənˌeɪt ˈæksəs]

Definition：Refer to the $x$-axis, the $y$-axis, or the $z$-axis.　坐标轴指 $x$ 轴、$y$ 轴或 $z$ 轴。

## Coordinate　坐标

n. [koʊˈɔrdənˌeɪt]

Definition：Refer to the $x$-coordinate, the $y$-coordinate, or the $z$-coordinate.　坐标指 $x$ 坐标、$y$ 坐标或 $z$ 坐标。

The coordinates of a point are represented by an ordered triple. 每个点的坐标都可以用有序三元组表示。

## Ordered Triple 有序三元组

n. [ˈɔːrdɚd ˈtrɪpəl]

Definition: The representation of the location (as a point) on the three-dimensional space. 有序三元组是点在三维空间上的位置的表达方法。

Notation: $(x, y, z)$, in which $x$ represents the $x$-coordinate of the point, $y$ represents the $y$-coordinate of the point, and $z$ represents the $z$-coordinate of the point. 在$(x, y, z)$中，$x$ 代表点的 $x$ 坐标，$y$ 代表点的 $y$ 坐标，$z$ 代表点的 $z$ 坐标。

If point $A$ is located at $(a, b, c)$, then we also say $A$ has coordinates of $(a, b, c)$. 若 $A$ 的位置在$(a, b, c)$，即 $A$ 的坐标为$(a, b, c)$。

Properties: 1. Each ordered triple $(a, b, c)$ on the coordinate plane is located in the intersection of the planes $x = a$, $y = b$, and $z = c$. 三维空间上的有序三元组$(a, b, c)$的位置为平面上 $x = a$、$y = b$、$z = c$ 的交点。

2. Points along the $x$-axis have $y$-coordinate and $z$-coordinate equal to 0. 沿着 $x$ 轴的点的 $y$ 坐标与 $z$ 坐标均为 0。

3. Points along the $y$-axis have $x$-coordinate and $z$-coordinate equal to 0. 沿着 $y$ 轴的点的 $x$ 坐标与 $z$ 坐标均为 0。

4. Points along the $z$-axis have $x$-coordinate and $y$-coordinate equal to 0. 沿着 $z$ 轴的点的 $x$ 坐标与 $y$ 坐标均为 0。

5. The origin's has coordinates of $(0, 0, 0)$. 坐标原点的坐标为$(0, 0, 0)$。

6. Figure 23-5 shows how to plot the blue point $(a, b, c)$. 图 23-5 展示了如何画出蓝色的点$(a, b, c)$。

We first locate the point $(a, b, 0)$ from the $xy$-plane, as shown in the gray rectangle. 首先须从 $xy$ 平面找到点$(a, b, 0)$，如灰色矩形所示。

We finish by moving the point $(a, b, 0)$ $c$ units up, as shown in the rectangle with vertices $(0, 0, 0)$, $(a, b, 0)$, $(a, b, c)$, $(0, 0, c)$. 最后把$(a, b, 0)$上移 $c$ 格，如顶点为$(0, 0, 0)$、$(a, b, 0)$、$(a, b, c)$、$(0, 0, c)$的矩形所示。

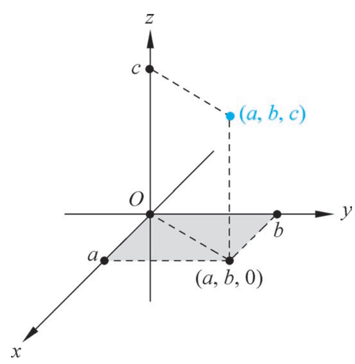


Figure 23-5

## Point 点

n. [pɔɪnt]

Definition: The visual representation on the three-dimensional space of an ordered triple. 点为有序三元组在三维空间上的视觉表示法。

## Origin 坐标原点

n. [ˈɔrədʒɪn]

Definition: The intersection of the coordinate axes. The origin has $(0, 0, 0)$ as the coordinates. 坐标原点是指坐标轴的交点，其坐标为$(0, 0, 0)$。

### Intercept　截距

n. [ˌɪntər'sept]

Definition：The number on a coordinate axis in which the graph intersects that axis.　截距是指图像与坐标轴相交的坐标数字。

For a graph in the three-dimensional space：

对于在三维空间的图像有：

The $x$-intercept is the $x$-coordinate of the point that the graph intersects the $x$-axis. At this point, the $y$ and $z$-coordinates are 0.

$x$ 截距是图像与 $x$ 轴交点的 $x$ 坐标。在这点上，$y$ 和 $z$ 坐标均为 0。

The $y$-intercept is the $y$-coordinate of the point that the graph intersects the $y$-axis. At this point, the $x$ and $z$-coordinates are 0.

$y$ 截距是图像与 $y$ 轴交点的 $y$ 坐标。在这点上，$x$ 和 $z$ 坐标均为 0。

The $z$-intercept is the $z$-coordinate of the point that the graph intersects the $z$-axis. At this point, the $x$ and $y$-coordinates are 0.

$z$ 截距是图像与 $z$ 轴交点的 $z$ 坐标。在这点上，$x$ 和 $y$ 坐标均为 0。

Phrases：$x$-intercept, $y$-intercept, $z$-intercept

Questions：What are the intercepts of the plane $5x+3y-2z=30$?

Answers：To find the $x$-intercept, set $y$ and $z$ to 0：

$$5x+3(0)-2(0)=30$$
$$5x=30$$
$$x=6.\quad \text{The } x\text{-intercept is } 6.$$

To find the $y$-intercept, set $x$ and $z$ to 0：

$$5(0)+3y-2(0)=30$$
$$3y=30$$
$$y=10.\quad \text{The } y\text{-intercept is } 10.$$

To find the $z$-intercept, set $x$ and $y$ to 0：

$$5(0)+3(0)-2z=30$$
$$-2z=30$$
$$z=-15.\quad \text{The } z\text{-intercept is } -15.$$

## 23.2　Formulas　公式

### Midpoint Formula　中点公式

n. ['mɪdˌpɔɪnt 'fɔrmjələ]

Definition：Given two points $A$ and $B$, the Midpoint Formula gives the midpoint of $\overline{AB}$.　中点公式给出端点为两个点的线段的中点。

Notation：Suppose point $A$ has the coordinates $(x_1, y_1, z_1)$ and point $B$ has the coordinates $(x_2, y_2, z_2)$, the Midpoint Formula is $M=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$, for which $AM=BM$.

Properties：1. Note that the $x$, $y$ and $z$ coordinates of $M$ are independent of each other.　计算中点

时,坐标之间相互独立。

2. Switching points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ leads to the same result. This is due to the Commutative Property of Addition. 根据加法交换律,把两点互换后代入公式,结果相同。

Questions: What is the coordinates of the midpoint of $\overline{AB}$ for each of the following if:

(1) $A=(5,9,4)$ and $B=(3,1,8)$

(2) $A=(-7,-2,-10)$ and $B=(-11,-16,-4)$

Answers: (1) $\left(\frac{5+3}{2},\frac{9+1}{2},\frac{4+8}{2}\right)=(4,5,6)$.

(2) $\left(\frac{-7+(-11)}{2},\frac{-2+(-16)}{2},\frac{-10+(-4)}{2}\right)=(-9,-9,-7)$.

## Distance Formula 距离公式

n. [ˈdɪstəns ˈfɔrmjələ]

Definition: Given two points $A$ and $B$, the Distance Formula gives the length of $\overline{AB}$. 距离公式给出端点为两个点的线段的长度。

Notation: Suppose point $A$ has coordinates $(x_1, y_1, z_1)$ and point $B$ has coordinates $(x_2, y_2, z_2)$, the Distance Formula is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$. Sometimes it is also written in $\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$, for which $\Delta x$ denotes the change in $x$, $\Delta y$ denotes the change in $y$, and $\Delta z$ denotes the change in $z$. All of $\Delta x$, $\Delta y$, and $\Delta z$ are nonnegative.

Properties: 1. This formula is based on applying Pythagorean Theorem several times. 这个公式源于多次运用勾股定理。

For two points $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$ on the $xyz$-plane, we can construct a rectangular prism for which $\overline{AB}$ is the long diagonal, as shown in Figure 23-6.

给出 $xyz$ 平面的两点 $A$ 与 $B$,要求它们的距离,先画一个长方体,使得 $\overline{AB}$ 为对角线,如图 23-6 所示。

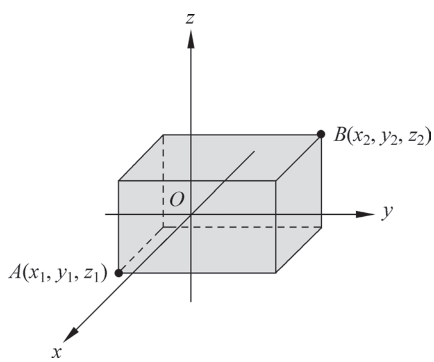

Figure 23-6

If exactly one of the coordinates are the same in $A$ and $B$, then the rectangular prism degenerates into a rectangle. This reduces to the Pythagorean Theorem on a two-dimensional plane.

若 $A$ 与 $B$ 的其中一个坐标相同,则长方体退化为矩形。距离可用勾股定理求出。

If exactly two of the coordinates are the same in $A$ and $B$, then the rectangular prism degenerates into a line segment parallel to the third axis. The distance is simply the difference between the third coordinate.

若 $A$ 与 $B$ 的其中两个坐标相同,则长方体退化为平行于第三个坐标轴的线段。距离为第三个坐标的差。

If all three of the coordinates are the same in $A$ and $B$, then the rectangular prism degenerates into a point, from which the distance is 0.
若 $A$ 与 $B$ 的所有坐标相同，则长方体退化为一个点，距离为 0。

As shown in Figure 23-7, we know that $AB^2 = AD^2 + BD^2$, which is $AB^2 = AD^2 + (\Delta z)^2$. By Pythagorean Theorem, $AD^2 = (\Delta x)^2 + (\Delta y)^2$. Therefore, $AB^2 = AD^2 + (\Delta z)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$, from which $AB = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.

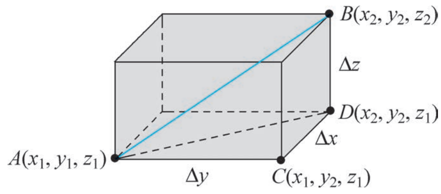

Figure 23-7

2. Switching points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ leads to the same result. This is due to the fact that the squares of a number and its opposite are equal: $a^2 = (-a)^2$. 把点互换后代入公式，结果相同，因为一个数的平方与它相反数的平方相等。

Note that the following formulas are equivalent.
以下公式等价：

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

$$AB = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2}.$$

$$AB = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

Questions: What is the distance for each of the following pairs of points?

(1) (1,3,8) and (2,6,3)

(2) (−5,−3,4) and (5,−8,−2)

Answers: (1) $\sqrt{(2-1)^2 + (6-3)^2 + (3-8)^2} = \sqrt{1^2 + 3^2 + (-5)^2} = \sqrt{35}$.

(2) $\sqrt{(5-(-5))^2 + (-8-(-3))^2 + (-2-4)^2} = \sqrt{10^2 + (-5)^2 + (-6)^2} = \sqrt{161}$.

# 23.3 Planes 平面

## Equation of a Plane 平面的方程式

Definition: The equation $ax + by + cz = d$, for which $a$, $b$, $c$, and $d$ are constants.

Properties: 1. Special cases of $a$, $b$, and $c$:

(1) If $a$ and $b$ are both 0, then we have $cz = d$, or $z = d/c$.
In other words, $z$ is a constant. This is a plane perpendicular to the $z$-axis.
当 $a$ 与 $b$ 均为 0，得出来的等式是 $cz = d$ 或 $z = d/c$，即 $z$ 等于一个常数。这个平面与 $z$ 轴垂直。

(2) If $a$ and $c$ are both 0, then we have $by = d$, or $y = d/b$.
In other words, $y$ is a constant. This is a plane perpendicular to the $y$-axis.
当 $a$ 与 $c$ 均为 0，得出来的等式是 $by = d$ 或 $y = d/b$，即 $y$ 等于一个常数。这个平面与 $y$ 轴垂直。

(3) If $b$ and $c$ are both 0, then we have $ax = d$, or $x = d/a$.
In other words, $x$ is a constant. This is a plane perpendicular to the $x$-axis.
当 $b$ 与 $c$ 均为 0，得出来的等式是 $ax = d$ 或 $x = d/a$，即 $x$ 等于一个常数。这个平

面与 $x$ 轴垂直。

2. Equation of planes that contain two coordinate axes:
包含两条坐标轴的平面的方程式:
(1) Equation of the $xy$-plane: $z=0$. ($a,b,d=0,c\neq 0$)
(2) Equation of the $xz$-plane: $y=0$. ($a,c,d=0,b\neq 0$)
(3) Equation of the $yz$-plane: $x=0$. ($b,c,d=0,a\neq 0$)

3. We can graph planes by intercepts:
可用截距画图。
To graph $ax+by+cz=d$, note the $x$-intercept is $\frac{d}{a}$, the $y$-intercept is $\frac{d}{b}$, and the $z$-intercept is $\frac{d}{c}$.
Locate the points $\left(\frac{d}{a},0,0\right)$, $\left(0,\frac{d}{b},0\right)$, and $\left(0,0,\frac{d}{c}\right)$ and draw a plane passing through these three points. Note that three non-collinear points determine a plane.
在方程式 $ax+by+cz=d$ 中, $x$ 截距为 $\frac{d}{a}$, $y$ 截距为 $\frac{d}{b}$, $z$ 截距为 $\frac{d}{c}$。
找出点 $\left(\frac{d}{a},0,0\right)$、$\left(0,\frac{d}{b},0\right)$、$\left(0,0,\frac{d}{c}\right)$, 画一个过这三点的平面。注意三个非共线点确定一个平面。

4. Coming up with an equation for a plane requires cross product of vectors, which will be covered in college calculus. 求出平面的等式需要用到向量积,这会在大学微积分课程提及。

Examples: To graph the plane $12x+15y+20z=60$, we locate the points $(5,0,0)$, $(0,4,0)$, and $(0,0,3)$, then draw a plane connecting these points, as shown in Figure 23-8.

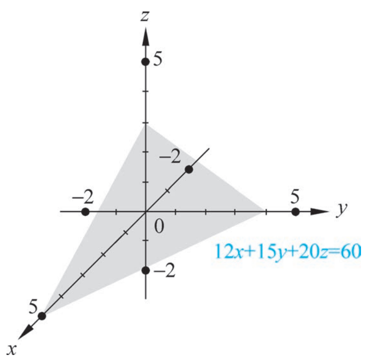

Figure 23-8

## 23.4 Systems of Linear Equations with Three Unknowns 三元一次方程组

Warning: Solving system of linear equations with three or more unknowns by graphing or substitution is extremely cumbersome. We will skip it and stick to elimination in the future.
说明:用画图法或代入消元法解三元或更多元一次方程组太复杂了,一般会忽略这两种方法,而用加减消元法,且以后都会用到。

视频 37

### Solving Systems of Linear Equations with Three Unknowns by Elimination 加减消元法

Definition: This is based on Solving Systems of Linear Equations with Two Unknowns by Elimination. 解三元一次方程组的加减消元法基于解二元一次方程的加减消元法。

Properties: Suppose you have a system of three linear equations: Equations $A$, $B$, and $C$. 假设现在有三元一次方程组:方程 $A$、$B$、$C$。