Dynamic Analysis of Control System in State Space

Preview

A modem complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner. To analyze such a system, it is essential to reduce the complexity of the mathematical expressions as well as to resort to computers for most of the tedious computations necessary in the analysis. The state-space approach to system analysis is best suited from this viewpoint.

This chapter and the next deal with the state-space analysis of control systems. Basic contexts of state-space analysis include dynamic analysis, controllability & observability. The former is presented in this chapter, while the latter in the next chapter. Basic design methods based on state-feedback control are given in Chapter 6.

Outline of the Chapter. Section 3. 1 presents the solution of the time-invariant homogeneous state equation. Section 3. 2 gives the properties of state-transition matrice. Section 3. 3 provides four calculation approaches of matrix exponential function. Section 3. 4 presents the solution of the time-invariant nonhomogeneous state equations. Section 3. 5 discusses the discretizing of linear time-invariant dynamic equation and its corresponding solution. Finally, computation of control system response with MATLAB is given in section 3. 6.

Desired Outcomes

Upon completion of Chapter 3, the following objectives should be achieved:

- Be capability of obtaining the solution of the state equations.
- Be able to calculate the state transition matrix through Laplace transform approach.
- Be aware of the basic properties of the state transition matrix.
- Understand discretization of linear time-invariant state differential equations.

3.1 Solving the Time-invariant Homogeneous State Equation

In this section, on the basis of the method of solution of the scalar differential equation, we shall obtain the general solution of the linear time-invariant state equation.

3.1.1 General solution of the scalar differential equation

Let us review the scalar differential equation.

$$\dot{x} = ax \tag{3.1}$$

In solving this equation, we assume a solution $x(t)$ of the form

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots \tag{3.2}$$

By substituting this assumed solution into Eq. (3. 1), we obtain

$$b_1+2b_2t+3b_3t^2+\cdots+kb_kt^{k-1}+\cdots=a(b_0+b_1t+b_2t^2+\cdots+b_kt^k+\cdots) \tag{3.3}$$

If the assumed solution is to be the true solution, Eq. (3.3) must hold for any t. Hence, equating the coefficents of the equal powers of t, we obtain

$$\begin{aligned} b_1 &= ab_0 \\ b_2 &= \frac{1}{2}ab_1=\frac{1}{2}a^2b_0 \\ b_3 &= \frac{1}{3}ab_2=\frac{1}{3\times 2}a^3b_0 \\ &\vdots \\ b_k &= \frac{1}{k!}a^kb_0 \end{aligned}$$

The value of b_0 is determined by substituting $t=0$ into Eq. (3.2), or

$$x(0)=b_0$$

Hence, the solution $x(t)$ can be written as

$$\begin{aligned} x(t) &= \left(1+at+\frac{1}{2!}a^2t^2+\cdots+\frac{1}{k!}a^kt^k+\cdots\right)x(0) \\ &= \mathrm{e}^{at}x(0) \end{aligned}$$

3.1.2 General solution of the vector-matrix differential equation

Let us now solve the vector-matrix differential equation

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x} \tag{3.4}$$

where $x=n$-vector, $\boldsymbol{A}=n\times n$ constant matrix.

By analogy with the scalar case, we assume that the solution is in the form of a vector power series in t, or

$$\boldsymbol{x}(t)=\boldsymbol{b}_0+\boldsymbol{b}_1t+\boldsymbol{b}_2t^2+\cdots+\boldsymbol{b}_kt^k+\cdots \tag{3.5}$$

By substituting this assumed solution into Eq. (3.4), we obtain

$$\boldsymbol{b}_1+2\boldsymbol{b}_2t+3\boldsymbol{b}_3t^2+\cdots+k\boldsymbol{b}_kt^{k-1}+\cdots=\boldsymbol{A}(\boldsymbol{b}_0+\boldsymbol{b}_1t+\boldsymbol{b}_2t^2+\cdots+\boldsymbol{b}_kt^k+\cdots) \tag{3.6}$$

If the assumed solution is to be the true solution, Eq. (3.6) must hold for any t. Hence, equating the coefficents of like powers of t both sides of Eq. (3.6), we obtain

$$\begin{aligned} \boldsymbol{b}_1 &= \boldsymbol{A}\boldsymbol{b}_0 \\ \boldsymbol{b}_2 &= \frac{1}{2}\boldsymbol{A}\boldsymbol{b}_1=\frac{1}{2}\boldsymbol{A}^2\boldsymbol{b}_0 \\ \boldsymbol{b}_3 &= \frac{1}{3}\boldsymbol{A}\boldsymbol{b}_2=\frac{1}{3\times 2}\boldsymbol{A}^3\boldsymbol{b}_0 \\ &\vdots \\ \boldsymbol{b}_k &= \frac{1}{k!}\boldsymbol{A}^k\boldsymbol{b}_0 \end{aligned}$$

By substituting $t=0$ into Eq. (3.5), we obtain

$$\boldsymbol{x}(0)=\boldsymbol{b}_0$$

Hence, the solution $\boldsymbol{x}(t)$ can be written as

$$\boldsymbol{x}(t)=\left(\boldsymbol{I}+\boldsymbol{A}t+\frac{1}{2!}\boldsymbol{A}^2t^2+\cdots+\frac{1}{k!}\boldsymbol{A}^kt^k+\cdots\right)\boldsymbol{x}(0)$$

The expression in the parentheses on the right-hand side of this last equation is an $n\times n$ marix. Because of its similarity to the infinite power series for a scalar exponential, we call it the **matrix exponential function** and write as

$$\boldsymbol{I}+\boldsymbol{A}t+\frac{1}{2!}\boldsymbol{A}^2t^2+\cdots+\frac{1}{k!}\boldsymbol{A}^kt^k+\cdots=\mathrm{e}^{\boldsymbol{A}t} \tag{3.7}$$

In terms of the matrix exponential function, the solution to Eq. (3.4) can be written as

$$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

Often, we represent the solution to Eq. (3.4) as

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)$$

where $n\times n$ matrix $\boldsymbol{\Phi}(t)$ is also called state-transition matrix. Obviously,

$$\boldsymbol{\Phi}(t)=\mathrm{e}^{\boldsymbol{A}t}$$

3.1.3 State-transition matrix

We can write the solution of the homogeneous state equation

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x} \tag{3.8}$$

as

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t)\boldsymbol{x}(0) \text{ or } \boldsymbol{x}(t)=\boldsymbol{\Phi}(t-t_0)\boldsymbol{x}(t_0) \tag{3.9}$$

From Eq. (3.9), we can see that the solution of Eq. (3.8) is simply a transformation of the initial condition, hence the name state-transition matrix, which contains all the information about the free motions of the system. $\boldsymbol{\Phi}(t)$ or $\boldsymbol{\Phi}(t-t_0)$ is not a constant matrix but a time-variant matrix, which makes $\boldsymbol{x}(0)$ or $\boldsymbol{x}(t_0)$ transfer to $\boldsymbol{x}(t)$ or $\boldsymbol{x}(t-t_0)$. The geometric significance of state trajectory, taking two-dimensional state vector for example, can be shown in Fig. 3.1.

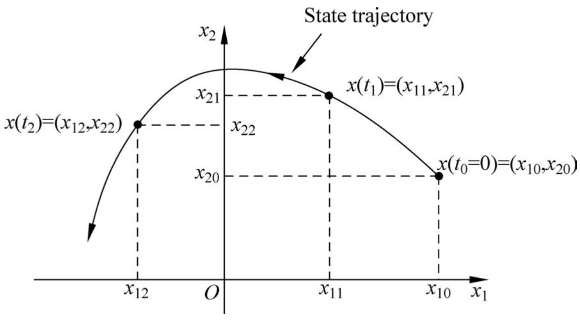

Fig. 3.1 The geometric significance of state trajectory

From Fig. 3.1, it can be seen that if the state $\boldsymbol{x}(t_0)=[x_{10}\quad x_{20}]^{\mathrm{T}}$ at initial time t_0 and $\boldsymbol{\Phi}(t_1)$ are known, the state $\boldsymbol{x}(t_1)=[x_{11}\quad x_{21}]^{\mathrm{T}}$ will be

$$\boldsymbol{x}(t_1)=\boldsymbol{\Phi}(t_1)\boldsymbol{x}(0) \tag{3.10}$$

And if $\boldsymbol{\Phi}(t_2)$ are known, the state $\boldsymbol{x}(t_2)=[x_{12} \quad x_{22}]^{\mathrm{T}}$ will be

$$\boldsymbol{x}(t_2)=\boldsymbol{\Phi}(t_2)\boldsymbol{x}(0) \tag{3.11}$$

Eq. (3.10) and Eq. (3.11) mean that initial state can be transferred to the state $\boldsymbol{x}(t_1)$ or $\boldsymbol{x}(t_2)$. If let the state $\boldsymbol{x}(t_1)$ be initial state, the state $\boldsymbol{x}(t_2)$ can be transferred from $\boldsymbol{x}(t_1)$. That is to say

$$\boldsymbol{x}(t_2)=\boldsymbol{\Phi}(t_2-t_1)\boldsymbol{x}(t_1) \tag{3.12}$$

Substituting $\boldsymbol{x}(t_1)$ of Eq. (3.10) into Eq. (3.12) can yield

$$\boldsymbol{x}(t_2)=\boldsymbol{\Phi}(t_2-t_1)\boldsymbol{\Phi}(t_1)\boldsymbol{x}(0) \tag{3.13}$$

Eq. (3.13) shows the state transformation from $\boldsymbol{x}(0)$ to $\boldsymbol{x}(t_1)$ first and the state transformation from $\boldsymbol{x}(t_1)$ to $\boldsymbol{x}(t_2)$ later.

Comparing Eq. (3.11) with Eq. (3.13), we have

$$\boldsymbol{\Phi}(t_2)=\boldsymbol{\Phi}(t_2-t_1)\boldsymbol{\Phi}(t_1)$$

or

$$\mathrm{e}^{\boldsymbol{A}t_2}=\mathrm{e}^{\boldsymbol{A}(t_2-t_1)}\mathrm{e}^{\boldsymbol{A}t_1}$$

Since state-transition matrix or matrix exponential function is very important in the state-space analysis of linear system, we shall next examine its properties.

3.2 Properties of State-transition Matrice

It is easy to notice that the state-transition matrix $\boldsymbol{\Phi}(t)$ is critical to the solution of the linear time-invariant state equation. Because $\boldsymbol{\Phi}(t)$ or $\mathrm{e}^{\boldsymbol{A}t}$ contains the complete information describing the system dynamic, it is important to understand its properties.

Some properties of the state-transition matrix $\boldsymbol{\Phi}(t)$ are summarized as following.

Property 1. $\boldsymbol{\Phi}(0)=\mathrm{e}^{\boldsymbol{A}0}=I$.

This property can be easily proved by substituting $t=0$ into Eq. (3.7).

Property 2. $\dot{\boldsymbol{\Phi}}(t)=\boldsymbol{A}\boldsymbol{\Phi}(t)=\boldsymbol{\Phi}(t)\boldsymbol{A}$.

Proof Because of convergence of the infinite series $\sum_{k=0}^{\infty}\boldsymbol{A}^k t^k/k!$, the series can be differentiated term by term to give

$$\begin{aligned}\dot{\boldsymbol{\Phi}}(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\boldsymbol{A}t}=\boldsymbol{A}+\boldsymbol{A}^2t+\frac{\boldsymbol{A}^3t^2}{2!}+\cdots+\frac{\boldsymbol{A}^kt^{k-1}}{(k-1)!}+\cdots\\&=\boldsymbol{A}\left[I+\boldsymbol{A}t+\frac{\boldsymbol{A}^2t^2}{2!}+\cdots+\frac{\boldsymbol{A}^{k-1}t^{k-1}}{(k-1)!}+\cdots\right]=\boldsymbol{A}\mathrm{e}^{\boldsymbol{A}t}=\boldsymbol{A}\boldsymbol{\Phi}(t)\\&=\left[\boldsymbol{I}+\boldsymbol{A}t+\frac{\boldsymbol{A}^2t^2}{2!}+\cdots+\frac{\boldsymbol{A}^{k-1}t^{k-1}}{(k-1)!}+\cdots\right]\boldsymbol{A}=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{A}=\boldsymbol{\Phi}(t)\boldsymbol{A}\end{aligned}$$

Property 3. $\boldsymbol{\Phi}(t_1\pm t_2)=\boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(\pm t_2)=\boldsymbol{\Phi}(\pm t_2)\boldsymbol{\Phi}(t_1)$.

Proof By substituting $t=t_1\pm t_2$ into Eq. (3.7), we have

$$\boldsymbol{\Phi}(t_1\pm t_2)=\mathrm{e}^{\boldsymbol{A}(t_1\pm t_2)}=\mathrm{e}^{\boldsymbol{A}t_1}\mathrm{e}^{\pm\boldsymbol{A}t_2}=\boldsymbol{\Phi}(t_1)\boldsymbol{\Phi}(\pm t_2)=\boldsymbol{\Phi}(\pm t_2)\boldsymbol{\Phi}(t_1)$$

where $\boldsymbol{\Phi}(t_1)$, $\boldsymbol{\Phi}(t_2)$, $\boldsymbol{\Phi}(t_1 \pm t_2)$ are the state transition matrice from $\boldsymbol{x}(0)$ to $\boldsymbol{x}(t_1)$, $\boldsymbol{x}(t_2)$, $\boldsymbol{x}(t_1 \pm t_2)$, respectively. This property shows that $\boldsymbol{\Phi}(t_1 \pm t_2)$ is equal to the product of $\boldsymbol{\Phi}(t_1)$ and $\boldsymbol{\Phi}(\pm t_2)$.

Property 4. $\boldsymbol{\Phi}^{-1}(t)=\boldsymbol{\Phi}(-t)$, $\boldsymbol{\Phi}^{-1}(-t)=\boldsymbol{\Phi}(t)$.

Proof By means of Property 3 above mentioned, we have

$$\boldsymbol{\Phi}(t-t)=\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(-t)=\boldsymbol{\Phi}(-t)\boldsymbol{\Phi}(t)=\boldsymbol{\Phi}(0)=\boldsymbol{I}$$

Thus, the Property 4 holds.

For linear time invariant system, $\boldsymbol{x}(t)=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)$ holds apparently, as a result, $\boldsymbol{x}(0)=\boldsymbol{\Phi}^{-1}(t)\boldsymbol{x}(t)=\boldsymbol{\Phi}(-t)\boldsymbol{x}(t)$ holds also.

Property 5. $\boldsymbol{\Phi}(t_2-t_1)\boldsymbol{\Phi}(t_1-t_0)=\boldsymbol{\Phi}(t_2-t_0)=\boldsymbol{\Phi}(t_1-t_0)\boldsymbol{\Phi}(t_2-t_1)$.

Proof $\boldsymbol{\Phi}(t_2-t_1)\boldsymbol{\Phi}(t_1-t_0)=\mathrm{e}^{\boldsymbol{A}(t_2-t_1)}\mathrm{e}^{\boldsymbol{A}(t_1-t_0)}=\mathrm{e}^{\boldsymbol{A}(t_2-t_0)}=\boldsymbol{\Phi}(t_2-t_0)$

Property 6. $[\boldsymbol{\Phi}(t)]^k=\boldsymbol{\Phi}(kt)$, k is positive integer.

Proof $[\boldsymbol{\Phi}(t)]^k=(\mathrm{e}^{\boldsymbol{A}t})^k=\mathrm{e}^{k\boldsymbol{A}t}=\mathrm{e}^{\boldsymbol{A}(kt)}=\boldsymbol{\Phi}(kt)$

Property 7. The matrix exponential has the property that

$$\mathrm{e}^{\boldsymbol{A}(s+t)}=\mathrm{e}^{\boldsymbol{A}s}\mathrm{e}^{\boldsymbol{A}t}$$

This can be proved as follows:

$$\begin{aligned}\mathrm{e}^{\boldsymbol{A}s}\mathrm{e}^{\boldsymbol{A}t}&=\left(\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k t^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{\boldsymbol{A}^k s^k}{k!}\right)=\sum_{k=0}^{\infty}\boldsymbol{A}^k\left(\sum_{i=0}^{\infty}\frac{t^i s^{k-i}}{i!(k-i)!}\right)\\&=\sum_{k=0}^{\infty}\boldsymbol{A}^k\ \frac{(t+s)^k}{k!}=\mathrm{e}^{\boldsymbol{A}(t+s)}\end{aligned}$$

In particular, if $s=-t$, then

$$\mathrm{e}^{\boldsymbol{A}t}\mathrm{e}^{-\boldsymbol{A}t}=\mathrm{e}^{-\boldsymbol{A}t}\mathrm{e}^{\boldsymbol{A}t}=\mathrm{e}^{\boldsymbol{A}(t-t)}=\boldsymbol{I}$$

Thus, the inverse of $\mathrm{e}^{\boldsymbol{A}t}$ is $\mathrm{e}^{-\boldsymbol{A}t}$. Since the inverse of $\mathrm{e}^{\boldsymbol{A}t}$ always exists, $\mathrm{e}^{\boldsymbol{A}t}$ is nonsingular.

Property 8. $\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B})t}=\mathrm{e}^{\boldsymbol{A}t}\mathrm{e}^{\boldsymbol{B}t}=\mathrm{e}^{\boldsymbol{B}t}\mathrm{e}^{\boldsymbol{A}t}$, if and only if $\boldsymbol{AB}=\boldsymbol{BA}$.

To prove this, note that

$$\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B})t}=\boldsymbol{I}+(\boldsymbol{A}+\boldsymbol{B})t+\frac{1}{2!}(\boldsymbol{A}+\boldsymbol{B})^2t^2+\frac{1}{3!}(\boldsymbol{A}+\boldsymbol{B})^3t^3+\cdots$$

$$\begin{aligned}\mathrm{e}^{\boldsymbol{A}t}\mathrm{e}^{\boldsymbol{B}t}&=\left(\boldsymbol{I}+\boldsymbol{A}t+\frac{1}{2!}\boldsymbol{A}^2t^2+\frac{1}{3!}\boldsymbol{A}^3t^3+\cdots\right)\left(\boldsymbol{I}+\boldsymbol{B}t+\frac{1}{2!}\boldsymbol{B}^2t^2+\frac{1}{3!}\boldsymbol{B}^3t^3+\cdots\right)\\&=\boldsymbol{I}+(\boldsymbol{A}+\boldsymbol{B})t+\frac{1}{2!}\boldsymbol{A}^2t^2+\boldsymbol{AB}t^2+\frac{1}{2!}\boldsymbol{B}^2t^2+\frac{1}{3!}\boldsymbol{A}^3t^3+\\&\quad\frac{1}{2!}\boldsymbol{A}^2\boldsymbol{B}t^3+\frac{1}{2!}\boldsymbol{AB}^2t^3+\frac{1}{3!}\boldsymbol{B}^3t^3+\cdots\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B})t}-\mathrm{e}^{\boldsymbol{A}t}\mathrm{e}^{\boldsymbol{B}t}=&\frac{\boldsymbol{BA}-\boldsymbol{AB}}{2!}t^2+\\&\frac{\boldsymbol{BA}^2+\boldsymbol{ABA}+\boldsymbol{B}^2\boldsymbol{A}+\boldsymbol{BAB}-2\boldsymbol{A}^2\boldsymbol{B}-2\boldsymbol{AB}^2}{3!}t^3+\cdots\end{aligned}$$

The difference between $\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B})t}$ and $\mathrm{e}^{\boldsymbol{A}t}\mathrm{e}^{\boldsymbol{B}t}$ vanishes if $\boldsymbol{A}$ and $\boldsymbol{B}$ commute.

It is very important to remember that

$$e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t} e^{\boldsymbol{B}t}, \quad \text{if } \boldsymbol{AB} = \boldsymbol{BA}$$

$$e^{(\boldsymbol{A}+\boldsymbol{B})t} \neq e^{\boldsymbol{A}t} e^{\boldsymbol{B}t}, \quad \text{if } \boldsymbol{AB} \neq \boldsymbol{BA}$$

3.3 Calculation of Matrix Exponential Function

In solving control engineering problems, it often becomes necessary to compute $e^{\boldsymbol{A}t}$. If matrix $e^{\boldsymbol{A}t}$ is given with all elements in numerical values, MATLAB provides a simple way to compute $e^{\boldsymbol{A}T}$, where T is a constant.

Aside from computational methods, several analytical methods are available for the computation of $e^{\boldsymbol{A}t}$. We shall present four methods here.

3.3.1 Direct calculation approach

The matrix exponential function $e^{\boldsymbol{A}t}$ can be calculated by using the infinite series in Eq. (3.7) as following

$$e^{\boldsymbol{A}t} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!}$$

It can be proved that the matrix exponential function of an $n \times n$ matrix $e^{\boldsymbol{A}t}$ converges absolutely for all finite t.

3.3.2 Laplace transform approach

Let us first consider the scalar case:

$$\dot{x} = ax \tag{3.14}$$

Taking the Laplace transform of Eq. (3.14), we obtain

$$sX(s) - x(0) = aX(s) \tag{3.15}$$

where $\boldsymbol{X}(s) = \boldsymbol{L}(\boldsymbol{x}(t))$.

Solving Eq. (3.15) for $X(s)$ gives

$$X(s) = \frac{x(0)}{s-a} = (s-a)^{-1} x(0)$$

The inverse Laplace transform of this last equation gives the solution

$$x(t) = e^{\boldsymbol{A}t} x(0)$$

The forgoing approach to the solution of the homogeneous scalar differential equation can be extended to the solution of the homogeneous state equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{3.16}$$

Taking the Laplace transform of Eq. (3.16), we obtain

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s)$$

where $\boldsymbol{X}(s) = \boldsymbol{L}(\boldsymbol{x}(t))$.

Hence

$$(s\boldsymbol{I}-\boldsymbol{A})\boldsymbol{X}(s)=\boldsymbol{x}(0)$$

Premultiplying both sides of this last equation by $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$, We obtain

$$\boldsymbol{X}(s)=(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{x}(0)$$

The inverse Laplace transform of $\boldsymbol{X}(s)$ gives the solution $\boldsymbol{x}$. Thus,

$$\boldsymbol{x}(t)=\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]\boldsymbol{x}(0) \tag{3.17}$$

Note that

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{\boldsymbol{I}}{s}+\frac{\boldsymbol{A}}{s^2}+\frac{\boldsymbol{A}^2}{s^3}+\cdots$$

Hence, inverse Laplace transform of $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$ gives

$$\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]=\boldsymbol{I}+\boldsymbol{A}t+\frac{\boldsymbol{A}^2t^2}{2!}+\frac{\boldsymbol{A}^3t^3}{3!}+\cdots=\mathrm{e}^{\boldsymbol{A}t} \tag{3.18}$$

From Eq. (3.17) and Eq. (3.18), the solution of Eq. (3.16) is obtained as

$$\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

The important of Eq. (3.18) lies in the fact that it provides a convenient means for finding the closed solution for the matrix exponential function.

In brief, the second method of computing $\mathrm{e}^{\boldsymbol{A}t}$ uses the Laplace transform approach. $\mathrm{e}^{\boldsymbol{A}t}$ can be given as follows:

$$\mathrm{e}^{\boldsymbol{A}t}=\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]$$

Thus, to obtain $\mathrm{e}^{\boldsymbol{A}t}$, first invert the matrix $(s\boldsymbol{I}-\boldsymbol{A})$. This results in a matrix whose elements are rational functions of s. Then take the inverse Laplace transform of each element of the matrix.

Example 3.1 Calculate the state-transition matrix $\mathrm{e}^{\boldsymbol{A}t}$ if $\boldsymbol{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$.

Solution. To find the state-transition matrix, let us first calculate the matrix $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$:

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}^{-1}=\frac{\operatorname{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}$$

$$=\frac{1}{(s+1)(s+2)}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}=\begin{bmatrix}\frac{2}{s+1}-\frac{1}{s+2} & \frac{1}{s+1}-\frac{1}{s+2}\\ -\frac{2}{s+1}+\frac{2}{s+2} & -\frac{1}{s+1}+\frac{2}{s+2}\end{bmatrix}$$

The state-transition matrix is the inverse Laplace transform of $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$, i. e.,

$$\mathrm{e}^{\boldsymbol{A}t}=\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]=\begin{bmatrix}2\mathrm{e}^{-t}-\mathrm{e}^{-2t} & \mathrm{e}^{-t}-\mathrm{e}^{-2t}\\ -2\mathrm{e}^{-t}+2\mathrm{e}^{-2t} & -\mathrm{e}^{-t}+2\mathrm{e}^{-2t}\end{bmatrix}$$

3.3.3 Linear transform approach

For a given square matrix $\boldsymbol{A}$, there exists a nonsingular transform matrix $\boldsymbol{P}$ such that $\bar{\boldsymbol{A}}=\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$. Then we have

$$
\begin{aligned}
e^{\bar{A}t} &= e^{P^{-1}APt} = I + P^{-1}APt + \frac{1}{2!}(P^{-1}AP)^2t^2 + \cdots + \frac{1}{k!}(P^{-1}AP)^k t^k + \cdots \\
&= P^{-1}\left(I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{k!}A^k t^k + \cdots\right)P = P^{-1}e^{At}P
\end{aligned}
$$

Thus

$$e^{At} = \boldsymbol{\Phi}(t) = P e^{P^{-1}APt} P^{-1}$$

The solution of $\boldsymbol{P}$ can be obtained by following equations

$$
\begin{cases}
Ap_i = \lambda_i p_i, & i = 1, 2, \cdots, n \\
P = [p_1 \quad p_2 \quad \cdots \quad p_n]
\end{cases}
$$

where p_i is a eigenvector corresponding eigenvalue λ_i.

Case 1: If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix $\boldsymbol{A}$ are distinct (matrix e^{At} can be transformed into a diagonal form), thus $\boldsymbol{\Phi}(t)$ (or e^{At}) can be given by (contain the n exponentials $e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$)

$$
\boldsymbol{\Phi}(t) = P e^{P^{-1}APt} P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}
$$

where $\boldsymbol{P}$ is a diagonalizing matrix for $\boldsymbol{A}$.

Proof Since

$$
P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}
$$

We have

$$
\begin{aligned}
e^{P^{-1}APt} &= \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} t \\
&\quad + \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} \lambda_1^3 & & & 0 \\ & \lambda_2^3 & & \\ & & \ddots & \\ 0 & & & \lambda_n^3 \end{bmatrix} \frac{t^3}{3!} + \cdots
\end{aligned}
$$

$$= \begin{bmatrix} 1+\lambda_1 t+\frac{\lambda_1^2 t^2}{2!}+\frac{\lambda_1^3 t^3}{3!}+\cdots & & & 0 \\ & 1+\lambda_2 t+\frac{\lambda_2^2 t^2}{2!}+\frac{\lambda_2^3 t^3}{3!}+\cdots & & \\ & & \ddots & \\ 0 & & & 1+\lambda_n t+\frac{\lambda_n^2 t^2}{2!}+\frac{\lambda_n^3 t^3}{3!}+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

According to $\boldsymbol{\Phi}(t)=\boldsymbol{P}e^{\boldsymbol{P}^{-1}\boldsymbol{APt}}\boldsymbol{P}^{-1}$, we can obtain

$$\boldsymbol{\Phi}(t)=\boldsymbol{P}\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}\boldsymbol{P}^{-1}$$

In particular, if the matrix $\boldsymbol{A}$ is diagonal, then

$$\boldsymbol{\Phi}(t)=e^{\boldsymbol{A}t}=\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

Particularly, if the matrix $\boldsymbol{A}$ is a companion one as following

$$\boldsymbol{A}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

Then linear transformation $\boldsymbol{P}$ can be obtained by use of the Vandermonde matrix as following

$$P=\begin{bmatrix}1 & 1 & \cdots & 1\\ \lambda_1 & \lambda_2 & \cdots & \lambda_n\\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}\end{bmatrix}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct eigenvalues of $\boldsymbol{A}$.

Example 3.2 Again consider the matrix given in Example 3.1, i.e. $\boldsymbol{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$. Calculate the state-transition matrix $e^{\boldsymbol{A}t}$ using linear transformation.

Solution. Since

$$\det(\lambda \boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix}\lambda & -1\\ 2 & \lambda+3\end{vmatrix}=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)$$

two distinct eigenvalues of $\boldsymbol{A}$ are $\lambda_1=-1$ and $\lambda_2=-2$. By inspection, the matrix $\boldsymbol{A}$ is a companion one, and linear transformation matrix is

$$\boldsymbol{P}=\begin{bmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{bmatrix}$$

The inverse of $\boldsymbol{P}$ is given by

$$\boldsymbol{P}^{-1}=\frac{\mathrm{adj}\boldsymbol{P}}{\det\boldsymbol{P}}=\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}$$

Hence diagonalizing matrix for $\boldsymbol{A}$ is now obtained as

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$$

Thus the state-transition matrix is

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{P}e^{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}t}\boldsymbol{P}^{-1}=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}\\ &= \begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\end{aligned}$$

When the eigenvalues of $\boldsymbol{A}$ are not all distinct, there are two cases as following.

Case 2: When a matrix $\boldsymbol{A}$ has multiple eigenvalues, the matrix $\boldsymbol{A}$ can also be diagonalized if the number of independent eigenvectors associated with each multiple-eigenvalues is equal to the multiplicity of the eigenvalues.

Example 3.3 Diagonalize the following matrix

$$\boldsymbol{A}=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Solution. The eigenvalues of $\boldsymbol{A}$ are $\lambda_1=\lambda_2=1, \lambda_3=2$. The eigenvectors associated with $\lambda_1=1$ can be found by solving the equation

$$(\lambda_1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{p}_i=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\boldsymbol{p}_i=0, \quad i=1,2$$

Note that two linearly independent eigenvectors $\boldsymbol{p}_1=[1 \quad 0 \quad 0]^{\mathrm{T}}$ and $\boldsymbol{p}_2=[0 \quad 1 \quad 0]^{\mathrm{T}}$ associated with $\lambda_1=\lambda_2=1$ can be found for this equation. Solving following equation

$$(\lambda_3\boldsymbol{I}-\boldsymbol{A})\boldsymbol{p}_3=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\boldsymbol{p}_3=0$$

The eigenvector associated with $\lambda_3=2$ can be found as $\boldsymbol{p}_3=[1 \quad 0 \quad -1]^{\mathrm{T}}$. Hence we obtain

$$\boldsymbol{P}=[\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \boldsymbol{p}_3]=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$\boldsymbol{P}^{-1}=\frac{\mathrm{adj}\boldsymbol{P}}{\det\boldsymbol{P}}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

The matrix $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is found to be

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

In this example, the matrix having repeated eigenvalues can be diagonalized. Generally, however, it is not always possible to diagonalize the matrix $\boldsymbol{A}$ having multiple-eigenvalues. In that case the matrix can be transformed only into a Jordan form by use of a linear transformation.

Case 3: When a matrix $\boldsymbol{A}$ has multiple eigenvalues, the number of independent eigenvectors associated with multiple-eigenvalue is often less than the multiplicity of the eigenvalue. In this case, the matrix $\boldsymbol{A}$ cannot be diagonalized, but it can be transformed into a Jordan canonical form. For example, if the eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_5, \cdots, \lambda_n$$

And there is only one independent eigenvector associated with multiple eigenvalue of order 3 at $\lambda_1=1$. Then matrix $\boldsymbol{A}$ can be transformed into a Jordan canonical form, and $\mathrm{e}^{\boldsymbol{A}t}$ can be given by

$$e^{\boldsymbol{A}t} = \boldsymbol{S} e^{\boldsymbol{J}t} \boldsymbol{S}^{-1}$$

where

$$\boldsymbol{J} = \left[\begin{array}{ccc|ccc} \lambda_1 & 1 & 0 & & & \\ & \lambda_1 & 1 & & & \\ & & \lambda_1 & & & \\ \hline & & & \lambda_4 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{array}\right]$$

Then transformation matrix $\boldsymbol{S}$ can be obtained by following equations

$$\begin{cases} \boldsymbol{A}\boldsymbol{s}_i = \lambda_i \boldsymbol{s}_i, \quad i = 4, 5, \cdots, n \\ \boldsymbol{s}_1 = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \end{bmatrix}^{\mathrm{T}} \\ \boldsymbol{S} = \begin{bmatrix} \boldsymbol{s}_1 & \dfrac{\mathrm{d}\boldsymbol{s}_1}{\mathrm{d}\lambda_1} & \dfrac{1}{2!} \cdot \dfrac{\mathrm{d}^2 \boldsymbol{s}_1}{\mathrm{d}\lambda_1^2} & \boldsymbol{s}_4 & \boldsymbol{s}_5 & \cdots & \boldsymbol{s}_n \end{bmatrix} \end{cases}$$

Suppose $\boldsymbol{J}$ is $n \times n$ Jordan canonical form

$$\boldsymbol{J} = \begin{bmatrix} \lambda_1 & 1 & \cdots & 0 \\ & \lambda_1 & 1 & \vdots \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix}$$

Then

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{J}t} = \begin{bmatrix} e^{\lambda_1 t} & t e^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2 & \cdots & t^{n-1} e^{\lambda_1 t}/(n-1)! \\ 0 & e^{\lambda_1 t} & t e^{\lambda_1 t} & \cdots & t^{n-2} e^{\lambda_1 t}/(n-2)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t e^{\lambda_1 t} \\ 0 & 0 & 0 & \cdots & e^{\lambda_1 t} \end{bmatrix}$$

From the above equation, it can be seen that $\boldsymbol{\Phi}(t)$ contains, in addition to the exponentials $e^{\lambda_1 t}$, terms like $t e^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \cdots, t^{n-1} e^{\lambda_1 t}$.

Case 4: When a matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Its eigenvalues are a pair of conjugate roots, i. e.

$$\lambda_{1,2} = \sigma \pm \mathrm{j}\omega$$

Then

$$e^{\boldsymbol{A}t}=\begin{bmatrix}\cos\omega t & \sin\omega t\\ -\sin\omega t & \cos\omega t\end{bmatrix}$$

Example 3.4 Transform the following matrix $\boldsymbol{A}$ into a Jordan canonical form.

$$\boldsymbol{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}$$

Solution. The characteristic equation of the matrix $\boldsymbol{A}$ is

$$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3=0$$

Thus, matrix $\boldsymbol{A}$ has a multiple eigenvalue of order 3 at $\lambda=1$. It can be shown that matrix $\boldsymbol{A}$ has a multiple eigenvector of order 3. The transformation matrix that will transform matrix $\boldsymbol{A}$ into a Jordan canonical form can be given by

$$\boldsymbol{S}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}$$

The inverse of matrix $\boldsymbol{S}$ is

$$\boldsymbol{S}^{-1}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}$$

Then it can be seen that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}=\boldsymbol{J}$$

Noting that

$$e^{\boldsymbol{J}t}=\begin{bmatrix}e^t & te^t & \frac{1}{2}t^2e^t\\ 0 & e^t & te^t\\ 0 & 0 & e^t\end{bmatrix}$$

We find

$$e^{\boldsymbol{A}t}=\boldsymbol{S}e^{\boldsymbol{J}t}\boldsymbol{S}^{-1}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}e^t & te^t & \frac{1}{2}t^2e^t\\ 0 & e^t & te^t\\ 0 & 0 & e^t\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}e^t-te^t+\frac{1}{2}t^2e^t & te^t-t^2e^t & \frac{1}{2}t^2e^t\\ \frac{1}{2}t^2e^t & e^t-te^t-t^2e^t & te^t+\frac{1}{2}t^2e^t\\ te^t+\frac{1}{2}t^2e^t & -3te^t-t^2e^t & e^t+2te^t+\frac{1}{2}t^2e^t\end{bmatrix}$$

3.3.4 Cayley-Hamilton Theorem

Cayley-Hamilton theorem is very useful in proving theorems involving matrix equations. In the following, we first present Cayley-Hamilton theorem, then give methods of computing $e^{\boldsymbol{A}t}$ based on Sylvester's interpolation as for two cases.

Considering an $n\times n$ matrix $\boldsymbol{A}$ and its characteristic equation:

$$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n=0$$

1. Cayley-Hamilton theorem

Every matrix $\boldsymbol{A}$ satisfies its own characteristic equation, or that

$$A^n+a_1A^{n-1}+\cdots+a_{n-1}A+a_nI=0 \tag{3.19}$$

Proof To prove this theorem, note that adj$(\lambda\boldsymbol{I}-\boldsymbol{A})$ is a polynomial in λ of degree $(n-1)$. That is,

$$\operatorname{adj}(\lambda\boldsymbol{I}-\boldsymbol{A})=\boldsymbol{B}_1\lambda^{n-1}+\boldsymbol{B}_2\lambda^{n-2}+\cdots+\boldsymbol{B}_{n-1}\lambda+\boldsymbol{B}_n$$

where $B_1=\boldsymbol{I}$. Since

$$(\lambda\boldsymbol{I}-\boldsymbol{A})\operatorname{adj}(\lambda\boldsymbol{I}-\boldsymbol{A})=[\operatorname{adj}(\lambda\boldsymbol{I}-\boldsymbol{A})](\lambda\boldsymbol{I}-\boldsymbol{A})=|\lambda\boldsymbol{I}-\boldsymbol{A}|\,\boldsymbol{I}$$

We obtain

$$\begin{aligned}|\lambda\boldsymbol{I}-\boldsymbol{A}|\,\boldsymbol{I}&=\boldsymbol{I}\lambda^n+a_1\boldsymbol{I}\lambda^{n-1}+\cdots+a_{n-1}\boldsymbol{I}\lambda+a_n\boldsymbol{I}\\&=(\lambda\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{B}_1\lambda^{n-1}+\boldsymbol{B}_2\lambda^{n-2}+\cdots+\boldsymbol{B}_{n-1}\lambda+\boldsymbol{B}_n)\\&=(\boldsymbol{B}_1\lambda^{n-1}+\boldsymbol{B}_2\lambda^{n-1}+\cdots+\boldsymbol{B}_{n-1}\lambda+\boldsymbol{B}_n)(\lambda\boldsymbol{I}-\boldsymbol{A})\end{aligned}$$

From this equation, we see that $\boldsymbol{A}$ and $\boldsymbol{B}_i$ $(i=1,2,\cdots,n)$ commute. Hence, the product of $(\lambda\boldsymbol{I}-\boldsymbol{A})$ and adj$(\lambda\boldsymbol{I}-\boldsymbol{A})$ becomes zero if either of these is zero. If $\boldsymbol{A}$ is substituted for λ in this last equation, then clearly $|\lambda\boldsymbol{I}-\boldsymbol{A}|$ becomes zero. Hence, we obtain

$$\boldsymbol{A}^n+a_1\boldsymbol{A}^{n-1}+\cdots+a_{n-1}\boldsymbol{A}+a_n\boldsymbol{I}=0$$

This proves the Cayley-Hamilton theorem, or Eq. (3.19).

2. Computation of $e^{\boldsymbol{A}t}$

The following method of computing $e^{\boldsymbol{A}t}$ is based on Sylvester's interpolation method. We shall first consider the case where the roots of the minimal polynomial of $\boldsymbol{A}$ are distinct. Then we shall deal with the case of multiple roots.

Case 1: Minimal polynomial of $\boldsymbol{A}$ involves only distinct roots. We shall assume that the degree of the minimal polynomial of $\boldsymbol{A}$ is m. by using Sylvester's interpolation formula, it can be shown that $e^{\boldsymbol{A}t}$ can be obtained by solving the following determinant equation:

$$\begin{vmatrix}1&\lambda_1&\lambda_1^2&\cdots&\lambda_1^{m-1}&e^{\lambda_1t}\\1&\lambda_2&\lambda_2^2&\cdots&\lambda_2^{m-1}&e^{\lambda_2t}\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\1&\lambda_{m-1}&\lambda_{m-1}^2&\cdots&\lambda_{m-1}^{m-1}&e^{\lambda_{m-1}t}\\1&\lambda_m&\lambda_m^2&\cdots&\lambda_m^{m-1}&e^{\lambda_mt}\\I&\boldsymbol{A}&\boldsymbol{A}^2&\cdots&\boldsymbol{A}^{m-1}&e^{\boldsymbol{A}t}\end{vmatrix}=0 \tag{3.20}$$

By solving Eq. (3.20) for $e^{\boldsymbol{A}t}$, $e^{\boldsymbol{A}t}$ can be obtained in terms of the $\boldsymbol{A}^k$ $(k=0,1,\cdots,m-1)$ and $e^{\lambda_i t}$ $(i=1,2,\cdots,m)$ (Eq. (3.20) may be expanded, for example, about the last column).

Notice that solving Eq. (3.20) for $e^{\boldsymbol{A}t}$ is the same as writing

$$e^{\boldsymbol{A}t}=\alpha_0(t)\boldsymbol{I}+\alpha_1(t)\boldsymbol{A}+\alpha_2(t)\boldsymbol{A}^2+\cdots+\alpha_{m-1}(t)\boldsymbol{A}^{m-1}$$

And determining the $\alpha_k(t)$ $(k=0,1,2,\cdots,m-1)$ by solving the following set of m equations for the $\alpha_k(t)$:

$$\begin{cases}\alpha_0(t)+\alpha_1(t)\lambda_1+\alpha_2(t)\lambda_1^2+\cdots+\alpha_{m-1}(t)\lambda_1^{m-1}=e^{\lambda_1 t}\\ \alpha_0(t)+\alpha_1(t)\lambda_2+\alpha_2(t)\lambda_2^2+\cdots+\alpha_{m-1}(t)\lambda_2^{m-1}=e^{\lambda_2 t}\\ \quad\vdots\\ \alpha_0(t)+\alpha_1(t)\lambda_m+\alpha_2(t)\lambda_m^2+\cdots+\alpha_{m-1}(t)\lambda_m^{m-1}=e^{\lambda_m t}\end{cases} \tag{3.21}$$

If $\boldsymbol{A}$ is an $n\times n$ matrix and has distinct eigenvalues, then the number of $\alpha_k(t)$'s to be determined is $m=n$. If $\boldsymbol{A}$ involves multiple eigenvalues but its minimal polynomial has only simple roots, however, then the number m of $\alpha_k(t)$'s to be determined is less than n.

Case 2: Minimal polynomial of $\boldsymbol{A}$ involves multiple roots. As an example, consider the case where the minimal polynomial of $\boldsymbol{A}$ has three equal roots $(\lambda_1=\lambda_2=\lambda_3)$ and has other roots $(\lambda_4,\lambda_5,\cdots,\lambda_m)$ that are all distinct. By using Sylvester's interpolation formula, it can be shown that $e^{\boldsymbol{A}t}$ can be obtained from the following determinant equation:

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & (m-1)(m-2)\lambda_1^{m-3}/2 & t^2e^{\lambda_1 t}/2\\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t}\\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t}\\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t}\\ I & A & A^2 & A^3 & \cdots & A^{m-1} & e^{\boldsymbol{A}t}\end{vmatrix}=0 \tag{3.22}$$

Eq. (3.22) can be solved for $e^{\boldsymbol{A}t}$ by expanding it about the last column.

It is noticed that, just as in Case 1, solving Eq. (3.22) for $e^{\boldsymbol{A}t}$ is the same as writing

$$e^{\boldsymbol{A}t}=\alpha_0(t)\boldsymbol{I}+\alpha_1(t)\boldsymbol{A}+\alpha_2(t)\boldsymbol{A}^2+\cdots+\alpha_{m-1}(t)\boldsymbol{A}^{m-1}$$

And determining the $\alpha_k(t)$ $(k=0,1,2,\cdots,m-1)$ by solving the following set of m equations for the $\alpha_k(t)$:

$$\begin{cases} \alpha_2(t) + 3\alpha_3(t)\lambda_1 + \cdots + (m-1)(m-2)\alpha_{m-1}(t)\lambda_1^{m-3}/2 = t^2 e^{\lambda_1 t}/2 \\ \alpha_1(t) + 2\alpha_2(t)\lambda_1 + 3\alpha_3(t)\lambda_1^2 + \cdots + (m-1)\alpha_{m-1}(t)\lambda_1^{m-2} = t e^{\lambda_1 t} \\ \alpha_0(t)I + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} = e^{\lambda_1 t} \\ \alpha_0(t)I + \alpha_1(t)\lambda_4 + \alpha_2(t)\lambda_4^2 + \cdots + \alpha_{m-1}(t)\lambda_4^{m-1} = e^{\lambda_4 t} \\ \quad \vdots \\ \alpha_0(t)I + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} = e^{\lambda_m t} \end{cases} \tag{3.23}$$

The extension to other cases where, for example, there are two or more sets of multiple roots will be apparent. Note that if the minimal polynomial of $\boldsymbol{A}$ is not found it is possible to substitute the characteristic polynomial. The number of computations may, of course, be increased.

Example 3.5 Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

From Eq. (3.20), we get

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ I & A & e^{\boldsymbol{A}t} \end{vmatrix} = 0$$

Substitute 0 for λ_1 and -2 for λ_2 in this last equation, we obtain

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ I & A & e^{\boldsymbol{A}t} \end{vmatrix} = 0$$

Expanding the determinant, we obtain

$$-2e^{\boldsymbol{A}t} + \boldsymbol{A} + 2\boldsymbol{I} - \boldsymbol{A}e^{-2t} = 0$$

or

$$e^{\boldsymbol{A}t} = \frac{1}{2}(\boldsymbol{A} + 2\boldsymbol{I} - \boldsymbol{A}e^{-2t}) = \frac{1}{2}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} e^{-2t} \right\}$$

$$= \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

An alternative approach is to use Eq. (3.21). We first determine $\alpha_0(t)$ and $\alpha_1(t)$ from

$$\alpha_0(t) + \alpha_1(t)\lambda_1 = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_2 = e^{\lambda_2 t}$$

Since $\lambda_1 = 0$ and $\lambda_2 = -2$, the last two equations become

$$\alpha_0(t)=1$$

$$\alpha_0(t)-2\alpha_1(t)=e^{-2t}$$

Solving for $\alpha_0(t)$ and $\alpha_1(t)$ gives

$$\alpha_0(t)=1,\quad \alpha_1(t)=\frac{1}{2}(1-e^{-2t})$$

Then $e^{\boldsymbol{A}t}$ can be written as

$$e^{\boldsymbol{A}t}=\alpha_0(t)\boldsymbol{I}+\alpha_1(t)\boldsymbol{A}=\boldsymbol{I}+\frac{1}{2}(1-e^{-2t})\boldsymbol{A}=\begin{bmatrix}1 & \frac{1}{2}(1-e^{-2t})\\ 0 & e^{-2t}\end{bmatrix}$$

3.4 Solution of Nonhomogeneous State Equations

In this section, the complete solution of nonhomogeneous state equation can be derived by two methods, respectively.

3.4.1 Direct method (or integral method)

We shall begin considering the scalar case

$$\dot{x}=ax+bu \tag{3.24}$$

Let us rewrite Eq. (3.24)

$$\dot{x}-ax=bu$$

Multiplying both sides of this equation by e^{-at}, we obtain

$$e^{-at}\left[\dot{x}(t)-ax(t)\right]=\frac{d}{dt}\left[e^{-at}x(t)\right]=e^{-at}bu(t)$$

Integrating this equation between 0 and t gives

$$e^{-at}x(t)=x(0)+\int_0^t e^{-a\tau}bu(\tau)d\tau$$

or

$$x(t)=e^{At}x(0)+e^{At}\int_0^t e^{-a\tau}bu(\tau)d\tau$$

The first term on the right-hand side is the response to the initial condition and the second term is the response to the input $u(t)$.

Let us now consider the nonhomogeneous state equation described by

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u} \tag{3.25}$$

where $x=n$-vector;

$u=r$-vector;

$\boldsymbol{A}=n\times n$ constant matrix;

$\boldsymbol{B}=n\times r$ constant matrix.

By writing Eq. (3.25) as

$$\dot{\boldsymbol{x}}(t)-\boldsymbol{A}\boldsymbol{x}(t)=\boldsymbol{B}\boldsymbol{u}(t)$$

And premultiplying both sides of this equation by $e^{-\boldsymbol{A}t}$, we obtain

$$e^{-\boldsymbol{A}t}\left[\dot{\boldsymbol{x}}(t)-\boldsymbol{A}\boldsymbol{x}(t)\right]=\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right]=e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

Integrating the preceding equation between 0 and t gives

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t)=\boldsymbol{x}(0)+\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

or

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\boldsymbol{x}(0)+e^{\boldsymbol{A}t}\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau \tag{3.26}$$

Eq. (3.26) can also be written as

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)+\int_0^t \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau \tag{3.27}$$

where $\boldsymbol{\Phi}(t)=e^{\boldsymbol{A}t}$. Eq. (3.26) and Eq. (3.27) is the solution of Eq. (3.25). The solution $\boldsymbol{x}(t)$ is clearly the sum of a term consisting of the transition of the initial state and a term arising from the input vector.

Thus far we have assumed the initial time to be zero. If, however, the initial time is given by t_0 instead of 0, then the solution to Eq. (3.27) must be modified to

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0)+\int_{t_0}^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

3.4.2 Laplace Transform Approach

The solution of nonhomogeneous state equations

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}$$

can also be obtained by the Laplace transform approach. The Laplace transform of this last equation yields

$$s\boldsymbol{X}(s)-\boldsymbol{x}(0)=\boldsymbol{A}\boldsymbol{X}(s)+\boldsymbol{B}\boldsymbol{U}(s)$$

or

$$(s\boldsymbol{I}-\boldsymbol{A})\boldsymbol{X}(s)=\boldsymbol{x}(0)+\boldsymbol{B}\boldsymbol{U}(s)$$

Premultiplying both sides of this last equation by $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$, we obtain

$$\boldsymbol{X}(s)=(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{x}(0)+(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$

Using the relationship given by $\boldsymbol{\Phi}(t)=e^{\boldsymbol{A}t}=\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]$, the inverse Laplace transform of this last equation can be obtained by use of the convolution integral as follows:

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\boldsymbol{x}(0)+\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

Example 3.6 Find the solution to the state equation given by

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}\boldsymbol{u}(t),\quad \boldsymbol{u}(t)=\boldsymbol{1}(t),\quad \boldsymbol{x}(0)=\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

Solution. (1) Using integral method. Since $\boldsymbol{u}(t)=\boldsymbol{1}(t)$, let $\xi=t-\tau$ yields

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)+\int_0^t \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)-\int_t^0 \boldsymbol{\Phi}(\xi)\boldsymbol{B}\mathrm{d}\xi$$

$$=\boldsymbol{\Phi}(t)\boldsymbol{x}(0)+\int_0^t \boldsymbol{\Phi}(\tau)\boldsymbol{B}\,\mathrm{d}\tau$$

From Example 3.2, we get

$$\boldsymbol{\Phi}(t)=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

Hence

$$\int_0^t \boldsymbol{\Phi}(\tau)\boldsymbol{B}\,\mathrm{d}\tau=\int_0^t \begin{bmatrix} 2e^{-\tau}-e^{-2\tau} & e^{-\tau}-e^{-2\tau} \\ -2e^{-\tau}+2e^{-2\tau} & -e^{-\tau}+2e^{-2\tau} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathrm{d}\tau=\int_0^t \begin{bmatrix} e^{-\tau}-e^{-2\tau} \\ -e^{-\tau}+2e^{-2\tau} \end{bmatrix}\mathrm{d}\tau$$

$$=\begin{bmatrix} -e^{-\tau}+\frac{1}{2}e^{-2\tau} \\ e^{-\tau}-e^{-2\tau} \end{bmatrix}\Bigg|_0^t=\begin{bmatrix} \frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix}$$

$$\boldsymbol{x}(t)=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}+\begin{bmatrix} \frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix}$$

(2) Using Laplace transform method. Again, from Example 3.2, we get

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\begin{bmatrix} \frac{2}{s+1}-\frac{1}{s+2} & \frac{1}{s+1}-\frac{1}{s+2} \\ -\frac{2}{s+1}+\frac{2}{s+2} & -\frac{1}{s+1}+\frac{2}{s+2} \end{bmatrix}$$

$$e^{\boldsymbol{A}t}=\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]=\begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

Thus

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)=\begin{bmatrix} \frac{2}{s+1}-\frac{1}{s+2} & \frac{1}{s+1}-\frac{1}{s+2} \\ -\frac{2}{s+1}+\frac{2}{s+2} & -\frac{1}{s+1}+\frac{2}{s+2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\frac{1}{s}$$

$$=\begin{bmatrix} \frac{1}{s+1}-\frac{1}{s+2} \\ -\frac{1}{s+1}+\frac{2}{s+2} \end{bmatrix}\cdot\frac{1}{s}$$

$$=\begin{bmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{2}\cdot\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}\cdot\frac{1}{s+2} \\ \frac{1}{s+1}-\frac{1}{s+2} \end{bmatrix}$$

Hence

$$\boldsymbol{x}(t)=\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\right]\boldsymbol{x}(0)+\boldsymbol{L}^{-1}\left[(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)\right]$$

$$= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix}$$

3.5 Solution of Discrete Nonhomogeneous State Equations

3.5.1 Discretization of linear time-invariant dynamic equation

Consider linear time-invariant dynamic equation

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}+\boldsymbol{D}\boldsymbol{u} \end{aligned} \tag{3.28}$$

Its solution is

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t-t_0)\boldsymbol{x}(t_0)+\int_{t_0}^{t}\boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau \tag{3.29}$$

The initial condition of the system of Eq. (3.28) is taken as

$$\boldsymbol{x}(t_0)\big|_{t_0=kT}=\boldsymbol{x}(kT)$$

where T is sampling time.

Eq. (3.29) becomes

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}(t-kT)\boldsymbol{x}(kT)+\int_{kT}^{t}\boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

Taking $t=(k+1)T$, we obtain

$$\boldsymbol{x}((k+1)T)=\boldsymbol{\Phi}(T)\boldsymbol{x}(kT)+\int_{kT}^{(k+1)T}\boldsymbol{\Phi}((k+1)T-\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau \tag{3.30}$$

when $kT\leqslant t\leqslant(k+1)T$, we have $\boldsymbol{u}(t)=\boldsymbol{u}(kT)$.

Then

$$\boldsymbol{x}((k+1)T)=\boldsymbol{\Phi}(T)\boldsymbol{x}(kT)+\left[\int_{kT}^{(k+1)T}\boldsymbol{\Phi}((k+1)T-\tau)\boldsymbol{B}\mathrm{d}\tau\right]\boldsymbol{u}(kT) \tag{3.31}$$

Let

$$\begin{aligned} \boldsymbol{G} &= \boldsymbol{\Phi}(T) \\ \boldsymbol{H} &= \int_{kT}^{(k+1)T}\boldsymbol{\Phi}((k+1)T-\tau)\boldsymbol{B}\mathrm{d}\tau=\int_{0}^{T}\boldsymbol{\Phi}(T-\tau)\boldsymbol{B}\mathrm{d}\tau=\int_{0}^{T}\boldsymbol{\Phi}(\tau)\boldsymbol{B}\mathrm{d}\tau \end{aligned} \tag{3.32}$$

Substituting Eq. (3.32) into Eq. (3.31) and omitting T yields the discretized state equation

$$\boldsymbol{x}(k+1)=\boldsymbol{G}\boldsymbol{x}(k)+\boldsymbol{H}\boldsymbol{u}(k) \tag{3.33}$$

As an algebraic equation, the discretized output equation is

$$\boldsymbol{y}(k)=\boldsymbol{C}\boldsymbol{x}(k)+\boldsymbol{D}\boldsymbol{u}(k) \tag{3.34}$$

Thus state-space representation of linear time-invariant discrete system is

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{G}\boldsymbol{x}(k)+\boldsymbol{H}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k)+\boldsymbol{D}\boldsymbol{u}(k) \end{aligned}$$

Example 3.7 Determine the discretization of the following continuous-time state equation.

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}\boldsymbol{u}(t)$$

It is assumed that $T=1\text{s}$.

Solution. From Example 3.2, we have

$$\mathrm{e}^{\boldsymbol{A}t}=\begin{bmatrix}2\mathrm{e}^{-t}-\mathrm{e}^{-2t} & \mathrm{e}^{-t}-\mathrm{e}^{-2t}\\ -2\mathrm{e}^{-t}+2\mathrm{e}^{-2t} & -\mathrm{e}^{-t}+2\mathrm{e}^{-2t}\end{bmatrix}$$

Hence

$$\boldsymbol{G}=\boldsymbol{\Phi}(T)=\mathrm{e}^{\boldsymbol{A}t}\big|_{t=T}=\begin{bmatrix}2\mathrm{e}^{-t}-\mathrm{e}^{-2t} & \mathrm{e}^{-t}-\mathrm{e}^{-2t}\\ -2\mathrm{e}^{-t}+2\mathrm{e}^{-2t} & -\mathrm{e}^{-t}+2\mathrm{e}^{-2t}\end{bmatrix}\Bigg|_{t=1}=\begin{bmatrix}0.6004 & 0.2325\\ -0.4651 & -0.0972\end{bmatrix}$$

$$\boldsymbol{H}=\int_0^T\boldsymbol{\Phi}(t)\boldsymbol{B}\,\mathrm{d}t=\int_0^1\begin{bmatrix}2\mathrm{e}^{-t}-\mathrm{e}^{-2t} & \mathrm{e}^{-t}-\mathrm{e}^{-2t}\\ -2\mathrm{e}^{-t}+2\mathrm{e}^{-2t} & -\mathrm{e}^{-t}+2\mathrm{e}^{-2t}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\mathrm{d}t$$

$$=\int_0^1\begin{bmatrix}\mathrm{e}^{-t}-\mathrm{e}^{-2t}\\ -\mathrm{e}^{-t}+2\mathrm{e}^{-2t}\end{bmatrix}\mathrm{d}t=\begin{bmatrix}\frac{1}{2}-\mathrm{e}^{-1}+\frac{1}{2}\mathrm{e}^{-2}\\ \mathrm{e}^{-1}-2\mathrm{e}^{-2}\end{bmatrix}=\begin{bmatrix}0.1998\\ 0.2325\end{bmatrix}$$

The resulting discretized state equation is

$$\boldsymbol{x}(k+1)=\begin{bmatrix}0.6004 & 0.2325\\ -0.4651 & -0.0972\end{bmatrix}\boldsymbol{x}(k)+\begin{bmatrix}0.1998\\ 0.2325\end{bmatrix}\boldsymbol{u}(k)$$

3.5.2 Approximation

If the instant of time T is sufficient small compared with the time constants of the system, then we have

$$\boldsymbol{G}=\boldsymbol{\Phi}(T)=\mathrm{e}^{\boldsymbol{A}T}=\boldsymbol{I}+\boldsymbol{A}T+\cdots\approx\boldsymbol{I}+\boldsymbol{A}T$$

$$\boldsymbol{H}=\int_0^T\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{B}\,\mathrm{d}t=\int_0^T\mathrm{e}^{\boldsymbol{A}t}\,\mathrm{d}t\cdot\boldsymbol{B}\approx\int_0^T(\boldsymbol{I}+\boldsymbol{A}t)\,\mathrm{d}t\cdot\boldsymbol{B}\approx T\boldsymbol{B}$$

Thus Eq. (3.33) can be rewritten approximately as

$$\boldsymbol{x}(k+1)=(\boldsymbol{I}+\boldsymbol{A}T)\boldsymbol{x}(k)+T\boldsymbol{B}\boldsymbol{u}(k)$$

3.5.3 Recursive algorithms of the discrete state equation

Consider discrete linear time-invariant state-space Eq. (3.33), state variable $\boldsymbol{x}(k)$ $(k=1,2,\cdots)$ is given recursively by Eq. (3.33)

$$\boldsymbol{x}(1)=\boldsymbol{G}\boldsymbol{x}(0)+\boldsymbol{H}\boldsymbol{u}(0)$$

$$\boldsymbol{x}(2)=\boldsymbol{G}\boldsymbol{x}(1)+\boldsymbol{H}\boldsymbol{u}(1)=\boldsymbol{G}^2\boldsymbol{x}(0)+\boldsymbol{G}\boldsymbol{H}\boldsymbol{u}(0)+\boldsymbol{H}\boldsymbol{u}(1)$$

$$\boldsymbol{x}(3)=\boldsymbol{G}\boldsymbol{x}(2)+\boldsymbol{H}\boldsymbol{u}(2)=\boldsymbol{G}^3\boldsymbol{x}(0)+\boldsymbol{G}^2\boldsymbol{H}\boldsymbol{u}(0)+\boldsymbol{G}\boldsymbol{H}\boldsymbol{u}(1)+\boldsymbol{H}\boldsymbol{u}(2)$$

$$\vdots$$

$$\boldsymbol{x}(k)=\boldsymbol{G}\boldsymbol{x}(k-1)+\boldsymbol{H}\boldsymbol{u}(k-1)$$

$$=\boldsymbol{G}^k\boldsymbol{x}(0)+\sum_{i=0}^{k-1}\boldsymbol{G}^{k-1-i}\boldsymbol{H}\boldsymbol{u}(i)\tag{3.35}$$

Substituting Eq. (3.35) into Eq. (3.34) yields

$$\begin{aligned}\boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \\ &= \boldsymbol{C}\boldsymbol{G}^k\boldsymbol{x}(0) + \boldsymbol{C}\sum_{i=0}^{k-1}\boldsymbol{G}^{k-1-i}\boldsymbol{H}\boldsymbol{u}(i) + \boldsymbol{D}\boldsymbol{u}(k)\end{aligned} \tag{3.36}$$

The solution $\boldsymbol{x}(k)$ is clearly the sum of a term consisting of the transition of the initial state and a term arising from the input vector (In other words, the solution consists of zero input response plus zero state response). On the other hand, it can be seen from Eq. (3.35) that as for the state response to control input, there is at least one-step delay, which is essential characteristics of discrete system.

Define state-transition matrix of discrete system as

$$\boldsymbol{\Phi}(k) = \boldsymbol{G}^k$$

Be similar to the state-transition matrix of continuous system, the state-transition matrix of discrete system have following properties:

$$\boldsymbol{\Phi}(k+1) = \boldsymbol{G}\boldsymbol{\Phi}(k)$$

$$\boldsymbol{\Phi}(0) = \boldsymbol{I}$$

$$\boldsymbol{\Phi}^{-1}(k) = \boldsymbol{\Phi}(-k)$$

$$\boldsymbol{\Phi}(k-k_2) = \boldsymbol{\Phi}(k-k_1)\boldsymbol{\Phi}(k_1-k_2), \quad k > k_1 > k_2$$

3.5.4 Z transform approach to the solution of the discrete state equation

The Z-transform of Eq. (3.33) yields

$$z\boldsymbol{X}(z) - z\boldsymbol{x}(0) = \boldsymbol{G}\boldsymbol{X}(z) + \boldsymbol{H}\boldsymbol{U}(z)$$

or

$$(z\boldsymbol{I} - \boldsymbol{G})\boldsymbol{X}(z) = z\boldsymbol{x}(0) + \boldsymbol{H}\boldsymbol{U}(z)$$

Premultiplying both sides of above equation by $(z\boldsymbol{I} - \boldsymbol{G})^{-1}$, we obtain

$$\boldsymbol{X}(z) = (z\boldsymbol{I} - \boldsymbol{G})^{-1}z\boldsymbol{x}(0) + (z\boldsymbol{I} - \boldsymbol{G})^{-1}\boldsymbol{H}\boldsymbol{U}(z)$$

The inverse Z transform of this last equation yields

$$\boldsymbol{x}(k) = Z^{-1}\left[(z\boldsymbol{I} - \boldsymbol{G})^{-1}z\right]\boldsymbol{x}(0) + Z^{-1}\left[(z\boldsymbol{I} - \boldsymbol{G})^{-1}\boldsymbol{H}\boldsymbol{U}(z)\right] \tag{3.37}$$

Comparing Eq. (3.35) with Eq. (3.37) yields

$$\boldsymbol{\Phi}(k) = \boldsymbol{G}^k = Z^{-1}\left[(z\boldsymbol{I} - \boldsymbol{G})^{-1}z\right] \tag{3.38}$$

3.6 Computation of Control System Response with MATLAB

3.6.1 Response to initial condition

Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{3.39}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \tag{3.40}$$

Take Laplace transforms of both sides of Eq. (3.39):

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) \tag{3.41}$$

Eq. (3.41) can be rewritten as

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{x}(0) \tag{3.42}$$

Taking the inverse Laplace transforms of Eq. (3.42), we obtain

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x}(0)\delta(t) \tag{3.43}$$

Now define

$$\dot{\boldsymbol{z}} = \boldsymbol{x} \tag{3.44}$$

Eq. (3.43) can be rewritten as

$$\ddot{\boldsymbol{z}} = \boldsymbol{A}\dot{\boldsymbol{z}} + \boldsymbol{x}(0)\delta(t) \tag{3.45}$$

Integrating Eq. (3.45) with respect to t, we obtain

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{x}(0)\boldsymbol{1}(t) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}u \tag{3.46}$$

where

$$\boldsymbol{B} = \boldsymbol{x}(0), \quad \boldsymbol{u} = \boldsymbol{1}(t)$$

From Eq. (3.44), the state $\boldsymbol{x}(t)$ is given by $\dot{\boldsymbol{z}}(t)$, thus

$$\boldsymbol{x} = \dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \tag{3.47}$$

The solution of Eq. (3.46) and Eq. (3.47) gives the response to the initial condition.

Summarizing, the response of Eq. (3.39) to the initial condition $x(0)$ is obtained by solving the state-space equations

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}$$

where

$$\boldsymbol{B} = x(0), \quad \boldsymbol{u} = \boldsymbol{1}(t)$$

Noting that $\boldsymbol{x} = \dot{\boldsymbol{z}}$, we can write Eq. (3.40) as

$$\boldsymbol{y} = \boldsymbol{C}\dot{\boldsymbol{z}} \tag{3.48}$$

Substituting Eq. (3.47) into Eq. (3.48), we obtain

$$\boldsymbol{y} = \boldsymbol{C}(\boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u}) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{z} + \boldsymbol{C}\boldsymbol{B}\boldsymbol{u} \tag{3.49}$$

The solution of Eq. (3.47) and Eq. (3.48) gives the response of the system to a given initial condition. MATLAB commands to obtain the response curves (output curves y1 versus t, y2 versus t, …, ym versus t) are as follows:

```
[y,z,t] = step(A,B,C*A,C*B,1,t);
y1= [1 0 0 … 0] * y';
y2= [0 1 0 … 0] * y';
⋮
ym= [0 0 0 … 1] * y';
plot(t,y1,t,y2,…,t,ym)
```

Example 3.8 Obtain the response of the following system subjected to the given initial condition:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

Obtaining the response of the system to the given initial condition becomes that of solving the unit-step response of the system

$$\dot{z} = \boldsymbol{A}z + \boldsymbol{B}u$$
$$\boldsymbol{x} = \boldsymbol{A}z + \boldsymbol{B}u$$

where

$$\boldsymbol{B} = \boldsymbol{x}(0), \quad u = 1(t)$$

A possible MATLAB program for obtaining the response is shown in MATLAB Program 3.1. The resulting response curves are plotted in Fig. 3.2.

3.6.2 Obtaining the response to an initial condition by use of the command initial

If the system is given in state-space form, then the command

initial(A,B,C,D,[initial condition],t)

will produce the response to the initial condition.

Suppose that we have the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \boldsymbol{D} = 0$$

$$\boldsymbol{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then the command initial can be used as shown in MATLAB Program 3.2 to obtain the response to the initial condition. The response curves $x_1(t)$ and $x_2(t)$ are shown in Fig. 3.3 and are the same as those shown in Fig. 3.2.

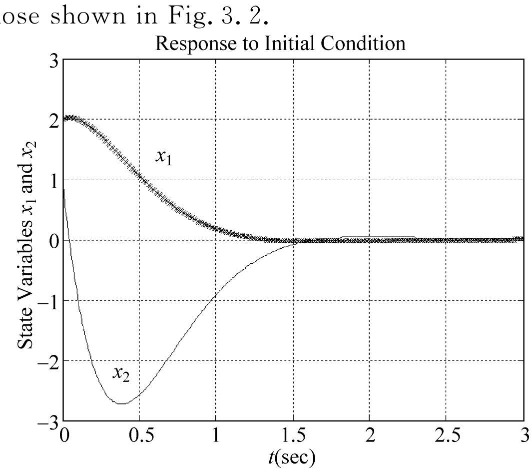

Fig. 3.2 Response of system in Example 3.8 to initial condition

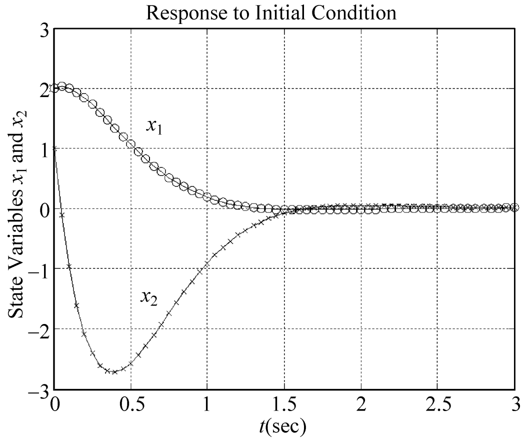

Fig. 3.3 Response curves to initial condition

MATLAB Program 3.1

```
>> t=0:0.01:3;
>> A=[0  1;-10  -5];
>> B=[2;1];
>>[x,z,t]=step (A,B,A,B,1,t);
>> x1= [1  0] * x';
>> x2= [0  1] * x';
>> plot(t,x1,'x',t,x2,'-')
>> grid
>> title('Response to Initial Condition')
>> xlabel('t Sec');
>> ylabel('State Variables x_1 and x_2');
>> gtext('x_1');
>> gtext('x_2');
```

MATLAB Program 3.2

```
>> t=0:0.05:3;
>> A=[0  1;-10  -5];
>> B=[0;0];
>> C=[0  0];
>> D=[0];
>>[y,x]=initial (A,B,C,D,[2;1],t);
>> x1= [1  0] * x';
>> x2= [0  1] * x';
>> plot(t,x1,'o',t,x2,'x');
>> grid
>> title('Response to Initial Condition');
>> xlabel('t Sec');
>> ylabel('State Variables x_1 and x_2');
>> gtext('x_1');
>> gtext('x_2');
```

Example 3.9 Consider the following system that is subjected to the given initial condition.

$$\dddot{y}+8\ddot{y}+17\dot{y}+10y=0$$

$$y(0)=2,\quad \dot{y}(0)=1,\quad \ddot{y}(0)=0.5$$

Assume that no external forcing function is present, and obtain the response $y(t)$ to the initial condition.

By defining the state variables as

$$x_1=y$$

$$x_2=\dot{y}$$

$$x_3=\ddot{y}$$

we obtain the following state-space representation of the system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-10&-17&-8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\quad \begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix}=\begin{bmatrix}2\\1\\0.5\end{bmatrix}$$

$$\boldsymbol{y}=\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

A possible MATLAB program to obtain the response $y(t)$ is given in MATLAB Program 3.3. The resulting response curve is shown in Fig. 3.4.

MATLAB Program 3.3

```
>> t=0:0.05:10;
>> A=[0  1  0;0  0  1;-10  -17  -8];
>> B=[0;0;0];
>> C=[1  0  0];
>> D=[0];
>> y=initial(A,B,C,D,[2;1;0.5],t);
>> plot(t,y);
>> grid
>> title('Response to Initial Condition')
>> xlabel('t (sec)');
>> ylabel('Output y');
```

Exercises

3.1 If matrix $\boldsymbol{A}$ is

$$\boldsymbol{A}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\-a_n&-a_{n-1}&-a_{n-2}&\cdots&a_1\end{bmatrix}$$

verify that the characteristic polynomial of $\boldsymbol{A}$ (Fig. 3.4) is

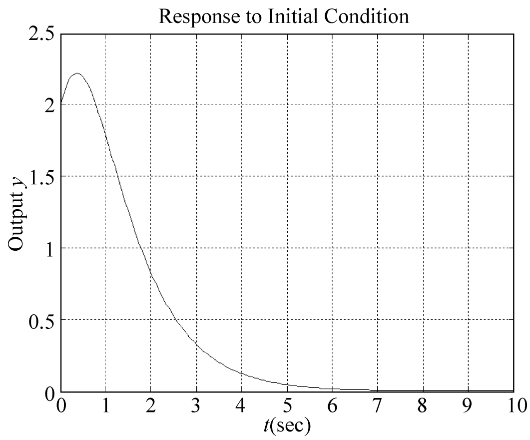

Fig. 3.4 Response $y(t)$ to the given initial condition

$$\Delta(\lambda)=\det(\lambda \boldsymbol{I}-\boldsymbol{A})=\lambda^{n}+a_{1}\lambda^{n-1}+a_{2}\lambda^{n-2}+\cdots+a_{n-1}\lambda+a_{n}$$

If λ_i is an eigenvalue of $\boldsymbol{A}$, try to verify that $\left[1,\ \lambda_i,\ \lambda_i^2,\ \cdots,\ \lambda_i^{n-1}\right]^{\mathrm{T}}$ is the eigenvectors corresponding to λ_i.

3.2 Given $\boldsymbol{A}=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$, try to get $\boldsymbol{A}^{101}$ and $\mathrm{e}^{\boldsymbol{A}t}$.

3.3 Given $\boldsymbol{A}$, $\boldsymbol{B}$ are constant square matrice, and $\boldsymbol{AB}=\boldsymbol{BA}$ Verify that the state transition matrix of $\dot{\boldsymbol{x}}=\mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{B}\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}$ is $\boldsymbol{\Phi}(t,t_0)=\mathrm{e}^{-\boldsymbol{A}t}\mathrm{e}^{(\boldsymbol{A}+\boldsymbol{B})(t-t_0)}\mathrm{e}^{\boldsymbol{A}t_0}$.

3.4 Calculate the resolvent matrice, i. e., $(s\boldsymbol{I}-\boldsymbol{A})^{-1}$, and the state transition matrice for the following matrice.

(1) $\boldsymbol{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

(2) $\boldsymbol{A}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

(3) $\boldsymbol{A}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 2\end{bmatrix}$

3.5 Transform the representations of the following systems into the Jordan canonical form.

(1) $\boldsymbol{A}=\begin{bmatrix}-2 & 1\\ -17 & -4\end{bmatrix}$, $\boldsymbol{b}=\begin{bmatrix}4\\ 8\end{bmatrix}$, $\boldsymbol{c}=\begin{bmatrix}0 & 1\end{bmatrix}$

(2) $\boldsymbol{A}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & 0\end{bmatrix}$, $\boldsymbol{b}=\begin{bmatrix}0\\ 0\\ 9\end{bmatrix}$, $\boldsymbol{c}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$

(3) $\boldsymbol{A}=\begin{bmatrix}-2 & 2 & -1\\ 0 & -2 & 0\\ 1 & -4 & 0\end{bmatrix}$, $\boldsymbol{b}=\begin{bmatrix}0\\ 0\\ 16\end{bmatrix}$, $\boldsymbol{c}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$

(4) $\boldsymbol{A}=\begin{bmatrix}0 & 1 & 0\\ 3 & 0 & 2\\ -12 & -7 & -6\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}2 & 3\\ 1 & 5\\ 7 & 1\end{bmatrix}$, $\boldsymbol{C}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

3.6 (1) Determine the characteristic equation, eigenvalues, and eigenvectors of the matrix

$$\boldsymbol{A}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}$$

(2) Transform the matrix given in (1) into the diagonal or Jordan form, finding the transformation matrix.

3.7 A linear system has the following state transition matrix

$$\boldsymbol{\Phi}(t)=\begin{bmatrix}2e^{-t}-e^{-2t} & -2e^{-t}+2e^{-2t}\\ e^{-t}-e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

Determine the system matrix $\boldsymbol{A}$ of the system.

3.8 Check whether the following matrice satisfy the conditions for the state transition matrix. If any matrix satisfies the condition, find the corresponding system matrix $\boldsymbol{A}$.

(1) $\boldsymbol{\Phi}(t)=\begin{bmatrix}2e^{-t}-e^{-2t} & -2e^{-t}+2e^{-2t}\\ e^{-t}-e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$

(2) $\boldsymbol{\Phi}(t)=\begin{bmatrix}1 & \dfrac{1-2e^{-2t}}{2}\\ 0 & e^{-2t}\end{bmatrix}$

(3) $\boldsymbol{\Phi}(t)=\begin{bmatrix}1 & 0 & 0\\ 0 & \sin t & \cos t\\ 0 & -\cos t & \sin t\end{bmatrix}$

(4) $\boldsymbol{\Phi}(t)=\begin{bmatrix}e^{-t} & 0 & 0\\ 0 & e^{-5t} & e^{-5t}\\ 0 & e^{-t} & e^{-5t}\end{bmatrix}$

3.9 The responses of a second-order system, $\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}$, to two different initial states are

$$\boldsymbol{x}(t)=\begin{bmatrix}e^{-3t}\\ -e^{-3t}\end{bmatrix}\ \text{when}\ \boldsymbol{x}(0)=\begin{bmatrix}1\\ -4\end{bmatrix}$$

$$\boldsymbol{x}(t)=\begin{bmatrix}e^{-2t}\\ -e^{-2t}\end{bmatrix}\ \text{when}\ \boldsymbol{x}(0)=\begin{bmatrix}2\\ -1\end{bmatrix}$$

respectively. Determine the system matrix $\boldsymbol{A}$.

3.10 Consider the state equation

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \boldsymbol{u}$$

Determine the state vector $\boldsymbol{x}(t)$ for $t \geqslant 0$ when the input $\boldsymbol{u}(t) = \boldsymbol{1}(t)$, using two different methods. It is assumed that the initial state is $\boldsymbol{x}(0) = [1 \quad 0]^{\mathrm{T}}$.

3.11 For a given system

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{-2t} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

try to compute the state response $\boldsymbol{x}(t)$ when $\boldsymbol{u}(t) = \boldsymbol{1}[t]$.

3.12 Given a state equation of a population emigration as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.01 \times (1-0.04) & 1.01 \times 0.02 \\ 1.01 \times 0.04 & 1.01 \times (1-0.02) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$x_1(0) = 10^7, \quad x_2(0) = 9 \times 10^7$$

Where $\boldsymbol{x}_1$ represents the city population and $\boldsymbol{x}_2$ represents the country population. If $k = 0$ represents year 1992, try to analyze the population distributing of the country and city from 1992 to 2010, and plot the corresponding distribution curve.

3.13 A discrete system is described by the difference equation.

$$y(k+2) + 3y(k+1) + 2y(k) = 2u(k+1) + 3u(k)$$

(1) Determine the state-space representation in controllable canonical form.

(2) Obtain the system response when the input

$$u(k) = \begin{cases} 1, & k = 0,1 \\ 0, & k \neq 0,1 \end{cases}$$

It is assumed that the initial state is zero.

3.14 Discretize the following continuous system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boldsymbol{y} = [1 \quad 0] \boldsymbol{x}$$

It is assumed that the sampling period is $T = 1\mathrm{s}$.